

Adelic resolution for homology sheaves

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Abstract

A generalization of the usual ideles group is proposed, namely, we construct certain adelic complexes for sheaves of K -groups on schemes. More generally, such complexes are defined for any abelian sheaf on a scheme. We focus on the case when the sheaf is associated to the presheaf of a homology theory with certain natural axioms, satisfied by K -theory. In this case it is proven that the adelic complex provides a flasque resolution for the above sheaf and that the natural morphism to the Gersten complex is a quasiisomorphism. The main advantage of the new adelic resolution is that it is contravariant and multiplicative in contrast to the Gersten resolution. In particular, this allows to reprove that the intersection in Chow groups coincides up to sign with the natural product in the corresponding K -cohomology groups. Also, we show that the Weil pairing can be expressed as a Massey triple product in K -cohomology groups with certain indices.

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1 Introduction

Classical adeles have been defined by A. Weil and C. Chevalley for a global field K in the following way: *adeles* are elements of the set $\mathbf{A}_K = \prod_v K_v$, where v runs over all

valuations of the field K , K_v is the completion of K at the point v , and the *restricted product* \prod means that we consider only collections $\{f_v\} \in \prod_v K_v$ such that $f_v \in \hat{\mathcal{O}}_v$

for almost all v , where $\hat{\mathcal{O}}_v \subset K_v$ is the complete local ring corresponding to v . The set of adeles has a natural structure of a topological ring. Its group of units is called the group of *ideles* and is equal to $\mathbf{A}_K^* = \prod_v K_v^*$, where the restricted product is taken in

the same sense as above with K_v replaced by K_v^* and $\hat{\mathcal{O}}_v$ replaced by $\hat{\mathcal{O}}_v^*$ (we omit the details about a topological structure of the ideles). The group of ideles plays a central role in the classical one-dimensional global class field theory. One of its main properties is the relation with the class group $Cl(K)$ of K . Actually there is a natural surjective homomorphism $\mathbf{A}_K^* \rightarrow Cl(K)$ defined by the formula $\{f_v\} \mapsto \sum_v \nu_v(f) \cdot [v]$, where the sum is taken over all non-archemidian valuations v and ν_v denotes a discrete valuation corresponding to v .

Further, J.-P. Serre used in [24] *non-complete* adeles on a *curve* X over a field k to prove the Riemann–Roch theorem for X . Namely he considered collections $\{f_x\} \in \prod_{x \in X} k(X)$ such that $f_x \in \mathcal{O}_{X,x}$ for almost all closed points $x \in X$, where $\mathcal{O}_{X,x}$ is the local ring of X at x . Similarly, one may construct a non-complete version of ideles. Moreover, Serre was using a certain complex of adeles though he did not mention this explicitly.

A. N. Parshin introduced in [19] non-complete (also called *rational*) adeles on a *surface*, and has constructed a certain *adelic complex*. In [20] there is also a multiplicative version of complete adeles on a surface, related to the K_2 -functor, and have been proposed a natural two-dimensional generalization of the classical class field theory. Later, A. A. Beilinson defined in [1] a complex of adeles $\mathbf{A}(X, \mathcal{O}_X)^\bullet$ for any Noetherian scheme X using simplicial language (in fact, the adelic complex is defined for any quasicoherent sheaf \mathcal{F} on X instead of \mathcal{O}_X). Let us describe explicitly the complexes of rational adeles

in low dimensions. For a curve X over a field k , it looks like

$$0 \rightarrow k(X) \oplus \prod_{x \in X} \mathcal{O}_{X,x} \rightarrow \prod_{x \in X} k(X) \rightarrow 0,$$

where the restricted product is taken in the above sense and the differential is defined using the natural inclusions by the formula $(f_X, \{f_x\}) \mapsto \{f_x - f_X\}$. For a surface X , the rational adelic complex has the following form:

$$0 \rightarrow k(X) \oplus \prod_{x \in X} \mathcal{O}_{X,x} \oplus \prod_{C \subset X} \mathcal{O}_{X,C} \rightarrow \prod_{C \subset X} k(X) \oplus \prod_{x \in X} k(X) \oplus \prod_{x \in C} \mathcal{O}_{X,C} \rightarrow \prod_{x \in C \subset X} k(X) \rightarrow 0,$$

where the restricted products can be defined explicitly in terms of the poles of functions, and the differentials are defined using the natural inclusion by the formulas

$$(f_X, \{f_x\}, \{f_C\}) \mapsto (\{f_C - f_X\}, \{f_x - f_X\}, \{f_x - f_C\})$$

and

$$(\{f_{XC}\}, \{f_{Xx}\}, \{f_{Cx}\}) \mapsto \{f_{Cx} - f_{Xx} + f_{XC}\}.$$

The Beilinson–Huber theorem tells that for any Noetherian scheme, the cohomology groups of the adelic complex $\mathbb{A}(X, \mathcal{O}_X)^\bullet$ are canonically isomorphic to the cohomology groups $H^i(X, \mathcal{O}_X)$, see [1], [14] (the analogous result holds true for any quasicoherent sheaf \mathcal{F} on X).

The aim of this paper is to give a version of these constructions for a class of sheaves of abelian groups on schemes different from quasicoherent sheaves, in particular, for *sheaves of K -groups*. Recall that a sheaf of K -groups $\mathcal{K}_n^X = \mathcal{K}_n(\mathcal{O}_X)$ is associated with the presheaf given by $U \mapsto K_n(k[U])$, $n \geq 0$, where $U \subset X$ is an open subset in the scheme X and $K_n(-)$ denotes the Quillen K -group. Recall that for a regular Noetherian separable scheme X of finite type over a field, there are Gersten (or Cousin) complexes $Gers(X, n)^\bullet$ whose cohomology groups are equal to $H^i(X, \mathcal{K}_n^X)$. On the other hand, for the case of a regular curve X , there is a natural quasiisomorphism of complexes

$$\begin{array}{ccccccc} 0 & \rightarrow & K_n(k(X)) \oplus \prod_{x \in C} K_n(\mathcal{O}_{X,x}) & \rightarrow & \prod_{x \in X} K_n(k(X)) & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & K_n(k(X)) & \rightarrow & \bigoplus_{x \in X} K_{n-1}(k(x)) & \rightarrow & 0 \end{array}$$

where the restricted product is taken in the same way as before for rational adeles with $k(X)$ replaced by $K_n(k(X))$ and $\mathcal{O}_{X,x}$ replaced by $K_n(\mathcal{O}_{X,x})$, $k(x)$ stands for the residue field at x , the complex on the bottom is the Gersten complex for the curve X , the first vertical morphism is the projection on the first summand, and the second one is given by residue maps. We give a higher-dimensional generalization of this. Recall that in general the Cousin complex of an abelian sheaf consists of direct sums over schematic points of fixed codimension. Following the general definition of adeles, we replace these direct sums by *adelic groups*, which are certain restricted products over *flags* of fixed

length, i.e., sequences of schematic points $\eta_0 \dots \eta_p$ on a scheme X such that $\eta_i \in \overline{\eta}_{i-1}$ for all $1 \leq i \leq p$. For the explicit construction of these groups in the simplest cases see Examples 2.32. Next, we construct an *adelic complex* $\mathbf{A}(X, \mathcal{F})^\bullet$, which consists of adelic groups. Under certain natural conditions, there is a canonical morphism of complexes $\nu_X : \mathbf{A}(X, \mathcal{F})^\bullet \rightarrow \text{Cous}(X, \mathcal{F})^\bullet$, where $\text{Cous}(X, \mathcal{F})^\bullet$ is the Cousin complex of \mathcal{F} on X . We restrict our attention to a special type of abelian sheaves, namely, sheaves associated with the presheaves of a homology theory that satisfies certain axioms (see Section 3.1). Our main result is that for such sheaves, the morphism ν_X is a quasiisomorphism on smooth varieties over an infinite perfect field. In particular, we get the following result.

Theorem 1.1. *There is a canonical morphism of complexes $\nu_X : \mathbf{A}(X, \mathcal{K}_n^X)^\bullet \rightarrow \text{Gers}(X, n)^\bullet$. This morphism is a quasiisomorphism if X is a smooth variety over an infinite perfect field.*

Recall that one of the main advantages of the Gersten complex is that it allows to relate explicitly cohomology of the sheaves of K -groups, called *K-cohomology*, with the (algebraic) geometry of X . In particular, the famous Bloch–Quillen formula says that $H^n(X, \mathcal{K}_n(\mathcal{O}_X)) = CH^n(X)$, see [22]. At the same time there is a canonical product between the sheaves of K -groups, induced by the product in K -groups themselves. This product structure cannot be prolonged to the Gersten complex: otherwise there would exist an intersection theory for algebraic cycles without taking them modulo rational equivalence.

The main advantage of the adelic construction is that the flag simplicial structure, involved in the definition of the adelic complexes, allows to define a product on them, i.e., the complex $\bigoplus_{n \geq 0} \mathbf{A}(X, \mathcal{K}_n^X)^\bullet$ is a DG-ring. Note that the general theory of sheaves provides many multiplicative simplicial resolutions of sheaves, i.e., resolutions carrying the product structure, for example, Čech or Godement resolutions. Nevertheless these resolutions seem to be too general to reflect the algebro-geometric structure of a scheme, for instance, relations of K -cohomology to algebraic cycles or direct images for proper morphisms. Though the adelic complex as presented here also does not have direct images, the covariant Gersten complex turns out to be a right DG-module over the DG-ring of adeles. Roughly speaking, the difference between the adelic complex $\mathbf{A}(X, \mathcal{K}_n^X)^\bullet$ and the Gersten complex $\text{Gers}(X, n)^\bullet$ consists of all possible *systems of local K-group equations along flags* for each irreducible subvariety on X . The main idea is that in order to get an intersection-product on the groups of algebraic cycles we enlarge them by systems of equations instead of taking them modulo rational equivalence.

Analysis of the adelic complex provides certain explicit formulas for products and also Massey higher products in K -cohomology. In particular, we obtain a new direct proof of the coincidence up to sign of the intersection product in Chow groups and the natural product in K -cohomology. Another example is the triple product $m_3(\alpha, l, \beta)$, where $\alpha \in CH^d(X)_l$, $\beta \in \text{Pic}^0(X)_l$. It occurs that this triple is equal to the Weil pairing of α and β . In the case of a curve the equality of the corresponding explicit adelic formula with the Weil pairing was proved by different methods in [13], [17], and [10].

The paper is organized as follows. First in Section 2.1 we define adelic groups for abelian sheaves on arbitrary schemes and study their basic properties, such as multi-

plicativity and contravariance. Section 2.2 shows an important relation of the adelic complex with the Cousin complex. In Sections 2.4 and 2.5 we establish more properties of adelic groups imposing rather mild conditions on sheaves. In Section 2.6 we define a new type of adelic groups, called \mathbf{A}' -adeles, which do not have a multiplicative structure but provide a flasque resolution for any Cohen–Macaulay sheaf.

In Section 3.1 we introduce the notion of a homology theory locally acyclic in fibrations (l.a.f. homology theory). We discuss the basic properties of an l.a.f. homology theory and define the associated homology sheaves. Then, in Section 3.2 we define strongly locally effaceable pairs of closed subvarieties on a smooth variety with respect to an l.a.f. homology theory. This is a globalization of the method from Quillen’s proof of the Gersten conjecture in the geometric case. Section 3.3 contains the existence and the addition results for strongly locally effaceable pairs. In particular, we get some uniform version of the local exactness of a Gersten resolution, which might have interest in its own right (see Corollary 3.20). In Section 3.4 we introduce the notion of patching systems of closed subvarieties on a smooth variety. This is our key tool for studying the relation between the adelic and the Gersten complexes. Section 3.5 contains the main result (Theorem 3.34): for any l.a.f. homology theory the adelic complex of the homology sheaves is quasiisomorphic to the Gersten complex on any smooth variety over an infinite perfect field. After several easy reductions the proof of the main theorem is reduced to a certain approximation result, namely Lemma 3.37, whose proof uses the developed technique of patching systems. Section 3.6 is devoted to the explicit construction of cocycles in the adelic complex representing cocycles in the Gersten complex.

Then we consider our main example of an l.a.f. homology theory, namely the K' -theory of schemes. We recall some general facts on K -sheaves and K -cohomology in Section 4.1. We also give some examples of the explicit K -adelic cocycles for algebraic cycles and we study the link between the K -adeles and the coherent differential adeles of Parshin and Beilinson. Section 4.2 provides an explicit construction of an Euler characteristic map from the K -groups of the exact category of complexes of coherent sheaves on a scheme T that are exact outside of a closed subscheme $S \subset T$ to the K' -groups of S . This map can be also constructed using R -spaces introduced in [3] or general properties of Waldhausen K -theory of perfect complexes given in [26]. Next, explicit formulas for products in K -cohomology in terms of Gersten cocycles are obtained in Section 4.3 as a consequence of the product structure on the adelic complex (Theorem 4.22). The case of certain Massey triple products is treated in Section 4.4. We also show the coincidence of the considered Massey triple product with the Weil pairing between zero-cycles and divisors (Proposition 4.27).

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2 Generalities on adeles

2.1 Definition and first properties

We use Beilinson's simplicial approach to higher-dimensional adeles, first defined by Parshin in the two-dimensional case (see [1], [19]). Besides, we follow most notations from [14]. For a cosimplicial group A^\bullet , let A^\bullet be the associated cochain complex. We define the differential in the tensor product $A^\bullet \otimes B^\bullet$ of two complexes A^\bullet and B^\bullet by the formula $d(a \otimes b) = da \otimes b + (-1)^{\deg a} a \otimes db$. For a scheme X , by $X^{(p)}$ denote the set of all schematic points of codimension p on X .

Let X be a scheme and let $P(X)$ be the set of all its schematic points. By $\bar{\eta}$ denote the closure of a point $\eta \in P(X)$. By definition, put

$$S(X)_p = \{(\eta_0, \eta_1, \dots, \eta_p) : \eta_i \in P(X), \eta_i \in \bar{\eta}_{i-1} \text{ for all } 1 \leq i \leq p\}.$$

An element $F = (\eta_p \dots \eta_0) \in S(X)_p$ is called a *flag of length p* . The assignment $X \mapsto S(X)_*$ is a covariant functor from the category of schemes to the category of simplicial sets. Let $\delta_i^p : S(X)_p \rightarrow S(X)_{p-1}$, $0 \leq i \leq p$ and $\sigma_i^p : S(X)_p \rightarrow S(X)_{p+1}$, $0 \leq i \leq p$ be the natural boundary and degeneracy maps, respectively.

There is an exact additive functor $\mathcal{F} \mapsto S(X, \mathcal{F})$ from the category of abelian sheaves on X to the category of cohomological abelian systems of coefficients on $S(X)_*$, given by $S(X, \mathcal{F})(\eta_0 \dots \eta_p) = \mathcal{F}_{\eta_0}$ for any flag $(\eta_0 \dots \eta_p) \in S(X)_p$. By $C(X, \mathcal{F})^*$ denote the cosimplicial group associated to the system of coefficients $S(X, \mathcal{F})$ on $S(X)_*$. We have $C(X, \mathcal{F})^p = \prod_{\eta_0 \dots \eta_p} \mathcal{F}_{\eta_0}$. Explicitly, the differential in the complex $C(X, \mathcal{F})^\bullet$ is given by

the formula $(df)_{\eta_0 \dots \eta_{p+1}} = \sum_{i=0}^p (-1)^i f_{\eta_0 \dots \hat{\eta}_i \dots \eta_{p+1}} \in F_{\eta_0}$ for any element $f \in C(X, \mathcal{F})^p$, where the hat means that we omit the corresponding element in the flag. By definition, put $C(M, \mathcal{F}) = \prod_{(\eta_0 \dots \eta_p) \in M} \mathcal{F}_{\eta_0}$ for a subset $M \subset S(X)_p$. In particular, we have $C(X, \mathcal{F})^p = C(S(X)_p, \mathcal{F})$.

Evidently, $S(X, \mathcal{F} \otimes \mathcal{G}) = S(X, \mathcal{F}) \otimes S(X, \mathcal{G})$ for any two sheaves \mathcal{F} and \mathcal{G} on X (we consider a point wise multiplication for systems of coefficients). Consequently there is a canonical morphism of complexes $C(X, \mathcal{F})^\bullet \otimes C(X, \mathcal{G})^\bullet \rightarrow C(X, \mathcal{F} \otimes \mathcal{G})^\bullet$ given by $(f \cdot g)_{\eta_0 \dots \eta_{p+q}} = f_{\eta_0 \dots \eta_p} \otimes g_{\eta_p \dots \eta_{p+q}}$ for any elements $f \in C(X, \mathcal{F})^p$, $g \in C(X, \mathcal{G})^q$.

If $f : X \rightarrow Y$ is a morphism of schemes, then there is a natural morphism of systems of coefficients $S(Y, f_* \mathcal{F}) \rightarrow f_* S(X, \mathcal{F})$ on $S(Y)_*$, where $f_* S(X, \mathcal{F})(G) = \prod_{F: f(F)=G} S(X, \mathcal{F})(F)$ for any flag $G \in S(Y)_*$. Consequently there is a morphism of cosimplicial groups $C(Y, f_* \mathcal{F})^* \rightarrow C(X, \mathcal{F})^*$.

Let \mathcal{F} be a sheaf of abelian groups on a scheme X . We put $\mathcal{F}_U = (i_U)_* i_U^* \mathcal{F}$ for any open embedding $i_U : U \hookrightarrow X$. Note that $(\mathcal{F}_U)_V = \mathcal{F}_{U \cap V}$ for two open subsets U and V in X . For a point $\eta \in X$, we put $[\mathcal{F}]_\eta = j_* j^*(\mathcal{F})$, where $j : \text{Spec } \mathcal{O}_{X, \eta} \rightarrow X$ is the natural morphism. We have $[\mathcal{F}]_\eta = \varinjlim \mathcal{F}_U$, where the limit is taken over all open subsets $U \subset X$ containing the point η .

For $M \subset S(X)_p$, $\eta \in P(X)$, by ${}_{\eta}M$ denote the following set:

$${}_{\eta}M = \{(\eta_1, \dots, \eta_p) \in S(X)_{p-1} : (\eta, \eta_1, \dots, \eta_p) \in M\}.$$

We define inductively the *adelic groups* $\mathbf{A}(M, \mathcal{F})$, $M \subset S(X)_p$ of \mathcal{F} on X in the following way.

Definition 2.1. For a subset $M \subset P(X) = S(X)_0$, we put

$$\mathbf{A}(M, \mathcal{F}) = C(M, \mathcal{F}) = \prod_{\eta \in M} \mathcal{F}_{\eta}.$$

For a subset $M \subset S(X)_p$, $p > 0$, we put

$$\mathbf{A}(M, \mathcal{F}) = \prod_{\eta \in P(X)} \tilde{\mathbf{A}}({}_{\eta}M, [\mathcal{F}]_{\eta}),$$

and

$$\tilde{\mathbf{A}}(M, [\mathcal{F}]_{\eta}) = \varinjlim_U \mathbf{A}(M, \mathcal{F}_U),$$

where the limit is taken over all open subsets $U \subset X$ containing the point η . Also, we put $\mathbf{A}(\emptyset, \mathcal{F}) = 0$. Elements of the adelic groups $\mathbf{A}(M, \mathcal{F})$ are called *adeles*.

Remark 2.2. The definition of adelic groups does not use the ring structure of the sheaf \mathcal{O}_X . In fact, all generalities about adeles that are discussed below make sense for abelian sheaves on any topological space such that every closed subset has a unique generic point.

Remark 2.3. This definition is analogous to the definition of adeles from [1]. The main differences with [1] are as follows: we replace coherent sheaves by sheaves of type \mathcal{F}_V and we use no completion in the construction. Consequently our adelic condition is more rough: if \mathcal{F} is a coherent sheaf on a Noetherian scheme X , then the defined above group $\mathbf{A}(M, \mathcal{F})$ contains the corresponding group of *rational adeles* (see [19] and [14]). However there is a comparison in the backward direction, see Proposition 4.7.

Remark 2.4. The scheme X is not included in our notation for adelic groups $\mathbf{A}(M, \mathcal{F})$. Nevertheless in what follows X could be always reconstructed from the notation for a sheaf \mathcal{F} .

Remark 2.5. It follows from the definition that for any subset $M \subset S(X)_p$, $p > 0$ and for any open subset $V \subset X$, we have

$$\mathbf{A}(M, \mathcal{F}_V) = \prod_{\eta \in P(X)} \varinjlim_{U_{\eta}} \mathbf{A}({}_{\eta}M, \mathcal{F}_{V \cap U_{\eta}}) = \varinjlim_{\{U_{\eta}\}} \prod_{\eta \in P(X)} \mathbf{A}({}_{\eta}M, \mathcal{F}_{V \cap U_{\eta}}),$$

where the second limit is taken over the set of systems $\{U_{\eta}\}$ of open subsets in X parameterized by schematic points η such that $\eta \in U_{\eta}$ for any $\eta \in P(X)$, and $\{U_{\eta}\} \leq \{U'_{\eta}\}$ if and only if $U_{\eta} \supseteq U'_{\eta}$ for all $\eta \in P(X)$.

Evidently, $\mathcal{F} \mapsto \mathbf{A}(M, \mathcal{F})$ is a covariant functor from the category of abelian sheaves on X to the category of abelian groups for any subset $M \subset S(X)_p$. It is easily shown that $\mathbf{A}((\eta_0 \dots \eta_p), \mathcal{F}) = \mathcal{F}_{\eta_0}$ for any element $(\eta_0 \dots \eta_p) \in S(X)_p$. For any subset $M \subset S(X)_p$, there is a natural morphism $\theta : \mathbf{A}(M, \mathcal{F}) \rightarrow C(M, \mathcal{F}) = \prod_{(\eta_0 \dots \eta_p) \in M} \mathcal{F}_{\eta_0}$.

Proposition 2.6.

- (i) For any subsets $M, N \subset S(X)_p$, $p \geq 0$ such that $M \cap N = \emptyset$, there is a decomposition $\mathbf{A}(M \cup N, \mathcal{F}) = \mathbf{A}(M, \mathcal{F}) \oplus \mathbf{A}(N, \mathcal{F})$;
- (ii) for any subset $M \subset S(X)_p$, $p > 0$, there are boundary maps $d_i^p : \mathbf{A}(\delta_i^{p+1}(M), \mathcal{F}) \rightarrow \mathbf{A}(M, \mathcal{F})$, $0 \leq i \leq p$;
- (iii) for any subset $M \subset S(X)_p$, there are isomorphisms $s_i^p : \mathbf{A}(\sigma_i^p(M), \mathcal{F}) \rightarrow \mathbf{A}(M, \mathcal{F})$, $0 \leq i \leq p$;
- (iv) for any subset $M \subset S(X)_p$ and for any sheaves \mathcal{F}, \mathcal{G} on X , there is a morphism $\mathbf{A}(M, \mathcal{F}) \otimes \mathbf{A}(M, \mathcal{G}) \rightarrow \mathbf{A}(M, \mathcal{F} \otimes \mathcal{G})$;
- (v) for any subset $M \subset S(Y)_p$ and for any morphism of schemes $f : X \rightarrow Y$, there is a natural morphism $f^* : \mathbf{A}(M, f_* \mathcal{F}) \rightarrow \mathbf{A}(f^{-1}(M), \mathcal{F})$;
- (vi) all the morphisms constructed in (i) – (v) commute via the map θ with their natural counterparts for the corresponding direct product groups.

Proof. The proof of (i) and the constructions in (ii), (iii) are the same as the proof of Propositions 2.1.5, Definition 2.2.2, and the proof of Proposition 2.3.1, respectively, from [14]. The only difference is that we use sheaves of type \mathcal{F}_U in place of coherent sheaves.

The proof of (iv) is by induction on p and uses that there is a natural morphism of sheaves $\mathcal{F}_U \otimes \mathcal{G}_V \rightarrow (\mathcal{F} \otimes \mathcal{G})_{U \cap V}$ for any open subsets $U, V \subset X$. Besides, for $p = 0$, the morphisms in question equals the point wise multiplication in stalks of sheaves.

The proof of (v) is also by induction on p and uses that there is a natural morphism of sheaves $(f_* \mathcal{F})_U \rightarrow f_*(\mathcal{F}_{f^{-1}(U)})$ for any open subset $U \subset Y$. Besides, for $p = 0$, the morphism in question equals the point wise map on stalks of sheaves.

The proof of (vi) is straightforward. \square

For a closed or an open subscheme $Y \subset X$, let i_Y be the corresponding embedding. For any subset $M \subset S(X)_p$, let $M(Y)$ be the set of flags on Y that are in M . The following consequences of Proposition 2.6 are needed for the sequel.

Corollary 2.7.

- (i) For any open subset $U \subset X$ and any subset $M \subset S(X)_p$, we have

$$\mathbf{A}(M(U), \mathcal{F}) = \mathbf{A}(M(U), i_U^* \mathcal{F}),$$

where the right hand side is the adelic group on U . In particular, $\mathbf{A}(M, \mathcal{F}) = \mathbf{A}(M(U), i_U^* \mathcal{F}) \oplus A$ for some subgroup $A \subset \mathbf{A}(M, \mathcal{F})$;

(ii) for any schematic point $\eta \in X$ and any subset $M \subset S(X)_p$, we have

$$\mathbf{A}(M(\eta), \mathcal{F}) = \mathbf{A}(M(\eta), j_\eta^* \mathcal{F}),$$

where $j_\eta : X_\eta = \text{Spec}(\mathcal{O}_{X,\eta}) \rightarrow X$ is the natural morphism of schemes, $M(\eta)$ is the set of flags on X_η that are from M , and the right hand side is the adelic group on X_η . In particular, $\mathbf{A}(M, \mathcal{F}) = \mathbf{A}(M(\eta), j_\eta^* \mathcal{F}) \oplus B$ for some subgroup $B \subset \mathbf{A}(M, \mathcal{F})$;

(iii) for any closed subset $Z \subset X$ and any subset $M \subset S(X)_p$, we have

$$\mathbf{A}(M(Z), \mathcal{F}) = \mathbf{A}(M(Z), i_Z^* \mathcal{F}),$$

where the right hand side is the adelic group on Z . In particular, for any sheaf \mathcal{G} on Z we have $\mathbf{A}(M(Z), (i_Z)_* \mathcal{G}) = \mathbf{A}(M(Z), \mathcal{G})$;

(iv) suppose that Y, Z are closed subschemes in X such that $X = Y \cup Z$; then for any sheaf \mathcal{F} on X and for any subset $M \subset S(X)_p$ we have

$$\mathbf{A}(M, \mathcal{F}) = [\mathbf{A}(M(Y), i_Y^* \mathcal{F}) \oplus \mathbf{A}(M(Z), i_Z^* \mathcal{F})] / \mathbf{A}(M(Y \cap Z), i_{Y \cap Z}^* \mathcal{F});$$

(v) consider a point $\eta \in X$ and a subset $M \subset S(X)_p$ such that any flag in M starts with η ; then for any sheaf \mathcal{F} on X and for any open subset $U \subset X$ containing η we have

$$\mathbf{A}(M, \mathcal{F}) = \mathbf{A}(M, \mathcal{F}_U).$$

We put $\mathbf{A}_s(X, \mathcal{F})^p = \mathbf{A}(S(X)_p, \mathcal{F})$. Using Proposition 2.6, we get the following statement.

Corollary 2.8.

- (i) There is a natural structure of a cosimplicial group on $\mathbf{A}_s(X, \mathcal{F})^*$ such that the natural morphism $\theta : \mathbf{A}_s(X, \mathcal{F})^* \rightarrow C(X, \mathcal{F})^*$ is a morphism of cosimplicial groups;
- (ii) for any two sheaves \mathcal{F} and \mathcal{G} on X there is a morphism of complexes $\mathbf{A}_s(X, \mathcal{F})^\bullet \otimes \mathbf{A}_s(X, \mathcal{G})^\bullet \rightarrow \mathbf{A}_s(X, \mathcal{F} \otimes \mathcal{G})^\bullet$, which commutes via θ with the morphism of complexes $C(X, \mathcal{F})^\bullet \otimes C(X, \mathcal{G})^\bullet \rightarrow C(X, \mathcal{F} \otimes \mathcal{G})^\bullet$;
- (iii) for any morphism of schemes $f : X \rightarrow Y$ and for any sheaf \mathcal{F} on X there is a morphism of cosimplicial groups $\mathbf{A}_s(Y, f_* \mathcal{F})^* \rightarrow \mathbf{A}_s(X, \mathcal{F})^*$, which commutes via θ with the morphism of cosimplicial groups $C(Y, f_* \mathcal{F})^* \rightarrow C(X, \mathcal{F})^*$.

For a cosimplicial group A^* , we put $A_{deg}^p = \sum_{i=0}^p \text{Im}(s_i^p)$, where $s_i^p : A^{p+1} \rightarrow A^p$ are the degeneracy maps; then A_{deg}^\bullet is a subcomplex in the complex A^\bullet (however there is no analogous inclusion of cosimplicial groups). It is well known that the quotient map $A^\bullet \rightarrow A^\bullet / A_{deg}^\bullet$ is a quasiisomorphism. We put $A_{red}^\bullet = A^\bullet / A_{deg}^\bullet$. Any morphism of simplicial groups $f : A^* \rightarrow B^*$ induces a morphism of complexes $f : A_{red}^\bullet \rightarrow B_{red}^\bullet$.

Definition 2.9. For a scheme X and an abelian sheaf \mathcal{F} on X , let the *adelic complex* $\mathbf{A}(X, \mathcal{F})^\bullet$ be $\mathbf{A}_s(X, \mathcal{F})_{red}^\bullet$.

It follows from Corollary 2.8 that for any sheaves \mathcal{F} and \mathcal{G} on X there is a morphism of complexes $\mathbf{A}(X, \mathcal{F})^\bullet \otimes \mathbf{A}(X, \mathcal{G})^\bullet \rightarrow \mathbf{A}(X, \mathcal{F} \otimes \mathcal{G})^\bullet$ and that for any morphism of schemes $f : X \rightarrow Y$ there is a morphism of complexes $f^* : \mathbf{A}(Y, f_* \mathcal{F}) \rightarrow \mathbf{A}(X, \mathcal{F})$. In particular, if \mathcal{A} is a sheaf of associative rings on X , then $\mathbf{A}(X, \mathcal{A})^\bullet$ is a DG-ring. Given a morphism of schemes $f : X \rightarrow Y$, we get a homomorphism of DG-rings $\mathbf{A}(Y, f_* \mathcal{A})^\bullet \rightarrow \mathbf{A}(X, \mathcal{A})^\bullet$. In addition, for any sheaf \mathcal{F} , there is a natural inclusion $\Gamma(X, \mathcal{F}) \hookrightarrow H^0(\mathbf{A}(X, \mathcal{F})^\bullet)$.

Suppose that the scheme X is Noetherian of finite dimension d . Then it is easily shown that

$$\begin{aligned} \mathbf{A}_s(X, \mathcal{F})^p &= \prod_{0 \leq i_0 \leq \dots \leq i_p \leq d} \mathbf{A}((i_0 \dots i_p), \mathcal{F}), \\ \mathbf{A}(X, \mathcal{F})^p &= \prod_{0 \leq i_0 < \dots < i_p \leq d} \mathbf{A}((i_0 \dots i_p), \mathcal{F}), \end{aligned}$$

where the expression $(i_0 \dots i_p)$ stands for the set of all flags $\eta_0 \dots \eta_p$ on X such that for any j , $0 \leq j \leq p$, we have $\text{codim}(\eta_j) = i_j$. We say that such flags are of type $(i_0 \dots i_p)$. For example, $\mathbf{A}(X, \mathcal{F})^0 = \prod_{0 \leq p \leq d} \mathbf{A}((p), \mathcal{F})$ and $\mathbf{A}((p), \mathcal{F}) = C((p), \mathcal{F}) = \prod_{\eta \in X^{(p)}} \mathcal{F}_\eta$. Thus

the adelic complex is bounded and has length d . In fact, the adelic complex is bounded for any Noetherian scheme X of finite Krull dimension by the maximal dimension of the irreducible components of X .

We may sheafify the above construction. Namely for any scheme X , an abelian sheaf \mathcal{F} on X , and a subset $M \subset S(X)_p$ there is an abelian presheaf $\underline{\mathbf{A}}(M, \mathcal{F})^*$ defined by the formula $U \mapsto \mathbf{A}_s(M(U), i_U^* \mathcal{F})$ (see Corollary 2.7(i)).

Proposition 2.10. *If the scheme X is Noetherian, then the presheaf $\underline{\mathbf{A}}_s(X, \mathcal{F})$ is actually a flasque sheaf.*

Proof. The flasqueness of this presheaf follows from Corollary 2.7(i). Clearly, it is enough to prove the sheaf property for the case of a finite open covering $\cup_\alpha V_\alpha$ of X . In this case we proceed by induction on p , using that for any collection of open subsets $U_\alpha \subset V_\alpha$ containing a fixed point $\eta \in X$, the open subset $\cap_\alpha U_\alpha \subset X$ also contains η . \square

Thus if X is Noetherian, then we get the flasque cosimplicial abelian sheaf $\underline{\mathbf{A}}_s(X, \mathcal{F})^*$ and the complexes of flasque sheaves $\underline{\mathbf{A}}_s(X, \mathcal{F})^\bullet$, $\underline{\mathbf{A}}_s(X, \mathcal{F})_{deg}^\bullet$, and $\underline{\mathbf{A}}(X, \mathcal{F})^\bullet$. Moreover, there is a morphism of complexes $\mathcal{F} \rightarrow \underline{\mathbf{A}}(X, \mathcal{F})^\bullet$, where \mathcal{F} is considered as a complex concentrated in the zero term.

Question 2.11. *Under which conditions on \mathcal{F} the complex of sheaves $\underline{\mathbf{A}}(X, \mathcal{F})^\bullet$ is a flasque resolution of \mathcal{F} ?*

Remark 2.12.

- (i) Let \mathcal{F}, \mathcal{G} be two sheaves on a Noetherian scheme X ; then the composition of the morphisms of complexes $\mathcal{F} \otimes \mathcal{G} \rightarrow \underline{\mathbf{A}}(X, \mathcal{F})^\bullet \otimes \underline{\mathbf{A}}(X, \mathcal{G})^\bullet \rightarrow \underline{\mathbf{A}}(X, \mathcal{F} \otimes \mathcal{G})^\bullet$ is the natural map described above.

- (ii) Let $f : X \rightarrow Y$ be a morphism of Noetherian schemes, \mathcal{F} be a sheaf on X ; then the composition of the morphisms of complexes $f_*\mathcal{F} \rightarrow \underline{\mathbf{A}}(Y, f_*\mathcal{F})^\bullet \rightarrow f_*\underline{\mathbf{A}}(X, \mathcal{F})^\bullet$ coincides with the value of the functor f_* at the natural morphism $\mathcal{F} \rightarrow \underline{\mathbf{A}}(X, \mathcal{F})^\bullet$ described above.

In particular, if \mathcal{A} is a sheaf of associative rings, then the map $\bigoplus_{i \geq 0} H^i(X, \mathcal{A}) \rightarrow \bigoplus_{i \geq 0} H^i(\mathbf{A}(X, \mathcal{F})^\bullet)$ is a homomorphism of rings.

We will use the following explicit description of adelic groups on Noetherian schemes. Let us introduce the following notation.

Definition 2.13. Let Z be a closed subscheme in a Noetherian scheme X and η be a schematic point in X ; then by $Z(\eta) \subset X^{(1)}$ denote the set of irreducible reduced divisors on X that are contained in Z and pass through η . Analogously, for a closed subset $W \subset X$, by $Z(W) \subset X^{(1)}$ denote the set of irreducible reduced divisors on X that are contained in Z and contain W .

Proposition 2.14. For any subset $M \subset S(X)_p$, we have

$$\mathbf{A}(M, \mathcal{F}) = \varinjlim_{\{D_{\eta_0 \dots \eta_k}\}} \prod_{(\eta_0 \dots \eta_p) \in M} (\mathcal{F}_{X \setminus D_{\eta_0 \dots \eta_{p-1}}})_{\eta_p},$$

where the limit is taken over the set of systems $\{D_{\eta_0 \dots \eta_k}\}$, $0 \leq k < p$ of effective, reduced, possibly reducible divisors on X parameterized by flags $\eta_0 \dots \eta_k$ that can be extended to the right to a flag from M with the following property. For any $k, 0 < k < p$ and for any “left part” $(\eta_0 \dots \eta_k)$ of a flag from M , we have

$$D_{\eta_0 \dots \eta_{k-1}}(\eta_k) \supseteq D_{\eta_0 \dots \eta_{k-1} \eta_k}(\eta_k) \quad (*)$$

and $D_{\eta_0}(\eta_0) = \emptyset$. The partial order on the systems $\{D_{\eta_0 \dots \eta_k}\}$, $0 \leq k < p$ is given by the flag wise embedding of divisors in X .

Proof. Using Remark 2.5, one proves by induction on p that

$$\mathbf{A}(M, \mathcal{F}) = \varinjlim_{\{U_{\eta_0 \dots \eta_k}\}} \prod_{(\eta_0 \dots \eta_p) \in M} (\mathcal{F}_{U_{\eta_0} \cap \dots \cap U_{\eta_0 \dots \eta_{p-1}}})_{\eta_p},$$

where the limit is taken over the set of systems $\{U_{\eta_0 \dots \eta_k}\}$, $0 \leq k < p$ of open subsets in X parameterized by “left parts” of flags from M such that for any “left part” $(\eta_0 \dots \eta_k)$ of a flag from M , we have $\eta_k \in U_{\eta_0 \dots \eta_k}$. The partial order on the systems $\{U_{\eta_0 \dots \eta_k}\}$, $0 \leq k < p$ is given as the inverse to the flag wise embedding of open subsets in X .

Since the scheme X is Noetherian, enlarging the complement $X \setminus U_{\eta_0 \dots \eta_k}$, we may assume that this complement is reduced, has pure codimension one, and does not contain η_k . Finally, we put $D_{\eta_0} = X \setminus U_{\eta_0}$ and $D_{\eta_0 \dots \eta_k} = D_{\eta_0 \dots \eta_{k-1}} \cup X \setminus U_{\eta_0 \dots \eta_k}$ for $1 \leq k < p$. \square

Claim 2.15. Condition $(*)$ implies that

$$D_{\eta_0 \dots \eta_k}(\eta_j) \supseteq D_{\eta_0 \dots \eta_k \dots \eta_{k+l}}(\eta_j)$$

for any $0 \leq j \leq k+1$, $0 \leq l \leq p-k-1$.

Proof. It is enough to show this for $l = 1$. Each irreducible divisor $D \in D_{\eta_0 \dots \eta_k, \eta_{k+1}}(\eta_j)$ contains η_{k+1} and thus belongs to $D_{\eta_0 \dots \eta_k}(\eta_{k+1})$. Moreover, since D contains η_j , we see that D also belongs to $D_{\eta_0 \dots \eta_k}(\eta_j)$. \square

2.2 Relation with the Cousin complex

Let X be a Noetherian catenary scheme such that all irreducible components of X have the same finite dimension d . For a closed subset $Z \subset X$ and a sheaf \mathcal{F} on X , denote by $\gamma_Z \mathcal{F}$ the sheaf on Z such that if $U \subset X$ is an open subset, then $(\gamma_Z \mathcal{F})(U \cap Z)$ consists of all sections in $\mathcal{F}(U)$ with support on $U \cap Z$. Thus, $R\gamma_Z = i_{Z!}^*$. One can also apply functors $R^p \gamma_Z$ to complexes of sheaves on X . Following the notations from [12], for any sheaf \mathcal{F} on X and any point $\eta \in X^{(p)}$, we put $H_\eta^p(X, \mathcal{F}) = (R^p \gamma_{\bar{\eta}} \mathcal{F})_\eta$. Let $\nu_{\eta\xi} : H_\eta^p(X, \mathcal{F}) \rightarrow H_\xi^{p+1}(X, \mathcal{F})$ be the natural map defined for any two points $\eta, \xi \in X$ such that $\xi \in \bar{\eta}$ and $\bar{\xi}$ has codimension one in $\bar{\eta}$ (see [12]). Let $\text{Cous}(X, \mathcal{F})^\bullet$ be the *Cousin complex* of \mathcal{F} on X , i.e.,

$$\text{Cous}(X, \mathcal{F})^p = \bigoplus_{\eta \in X^{(p)}} H_\eta^p(X, \mathcal{F}),$$

where the differential is the sum of the maps $\nu_{\eta\xi}$. Let us sheafify the Cousin complex, namely, consider the complex of sheaves $\underline{\text{Cous}}(X, \mathcal{F})^\bullet$ given by $\underline{\text{Cous}}(X, \mathcal{F})^p = \bigoplus_{\eta \in X^{(p)}} (i_{\bar{\eta}})_* H_\eta^p(X, \mathcal{F})$, where for each point $\eta \in X^{(p)}$ we consider $H_\eta^p(X, \mathcal{F})$ as a constant sheaf on $\bar{\eta}$. There is a natural map of complexes $\mathcal{F} \rightarrow \underline{\text{Cous}}(X, \mathcal{F})^\bullet$, where we consider \mathcal{F} as a complex concentrated in the zero term.

Let $0 \leq i_0 < \dots < i_p$ be a strictly increasing sequence of natural numbers; then the *depth* of $(i_0 \dots i_p)$ is the maximal natural number $l \geq 0$ such that $(i_0 \dots i_l) = (0 \dots l)$ if $i_0 = 0$. Otherwise, the depth of $(i_0 \dots i_p)$ equals -1 .

Proposition 2.16. *Let $(i_0 \dots i_p)$ be a strictly increasing sequence of natural numbers such that $i_p \leq d$. Suppose that the depth of $(i_0 \dots i_p)$ is $l \geq 0$; then there exists a natural map*

$$\nu_{0 \dots l} : \mathbf{A}((01 \dots li_{l+1} \dots i_p), \mathcal{F}) \rightarrow \bigoplus_{\eta \in X^{(l)}} \mathbf{A}((0(i_{l+1} - l) \dots (i_p - l)), R^l \gamma_{\bar{\eta}} \underline{\text{Cous}}(X, \mathcal{F})^\bullet),$$

where the right hand side is the direct sum of adelic groups on $\eta \in X^{(l)}$. Moreover, for any adele $f \in \mathbf{A}((01 \dots li_{l+1} \dots i_p), \mathcal{F})$ and any flag $\eta_l \eta_{l+1} \dots \eta_{i_p}$ of type $(li_{l+1} \dots i_p)$ on X , we have

$$\theta(\nu_{0 \dots l}(f))_{\eta_l \eta_{l+1} \dots \eta_{i_p}} = \sum_{\eta_0 \dots \eta_l} (\nu_{\eta_{l-1} \eta_l} \circ \dots \circ \nu_{\eta_0 \eta_1})(\theta(f)_{\eta_0 \dots \eta_l \eta_{l+1} \dots \eta_{i_p}}),$$

where the sum in the left hand side is actually finite.

Proof. The proof is by induction on l . For $l = 0$, by Proposition 2.7(iii), (iv), (v), we have the natural map

$$\mathbf{A}((0i_1 \dots i_p), \mathcal{F}) = \bigoplus_{\eta \in X^{(0)}} \mathbf{A}((0i_1 \dots i_p)(\bar{\eta}), i_{\bar{\eta}}^* \mathcal{F}) = \bigoplus_{\eta \in X^{(0)}} \mathbf{A}((0i_1 \dots i_p), \gamma_{\bar{\eta}} \mathcal{F}) \rightarrow$$

$$\rightarrow \bigoplus_{\eta \in X^{(0)}} \mathbf{A}((0i_1 \dots i_p), \gamma_{\bar{\eta}} \underline{Cous}(X, \mathcal{F})^\bullet).$$

Further, note that for the composition of closed embeddings $Z' \subset Z \subset X$, where Z has pure codimension l in X and Z' has pure codimension one in Z , the natural morphism of complexes

$$R^l \gamma_Z \underline{Cous}(X, \mathcal{F})^\bullet \rightarrow \gamma_Z \underline{Cous}(X, \mathcal{F})^\bullet[l]$$

induces the morphism of sheaves

$$R^1 \gamma_{Z'/Z} (R^l \gamma_Z \underline{Cous}(X, \mathcal{F})^\bullet) \rightarrow R^1 \gamma_{Z'/Z} (\gamma_Z \underline{Cous}(X, \mathcal{F})^\bullet[l]) = R^{l+1} \gamma_{Z'} \underline{Cous}(X, \mathcal{F})^\bullet.$$

Therefore, to prove the proposition by induction on l , it is enough to consider the case $l = 1$ and X is irreducible. Recall that for any closed subscheme $D \subset X$, there is a morphism of sheaves $\mathcal{F}_U \rightarrow (i_D)_* R^1 \gamma_D \mathcal{F}$, where $U = X \setminus D$. Hence, using the same argument as for the case $l = 0$, we get the map

$$\begin{aligned} \mathbf{A}((0i_2 \dots i_p), \mathcal{F}) &= \varinjlim_U \mathbf{A}((1i_2 \dots i_p), \mathcal{F}_U) \rightarrow \varinjlim_{D=X \setminus U} \mathbf{A}((1i_2 \dots i_p), (i_D)_* R^1 \gamma_D \mathcal{F}) = \\ &= \varinjlim_D \mathbf{A}(0(i_2 - 1) \dots (i_p - 1), R^1 \gamma_D \mathcal{F}) = \bigoplus_{\eta \in X^{(1)}} \mathbf{A}((0(i_2 - 1) \dots (i_p - 1)), R^1 \gamma_{\bar{\eta}} \mathcal{F}) \rightarrow \\ &\rightarrow \bigoplus_{\eta \in X^{(1)}} \mathbf{A}((0(i_2 - 1) \dots (i_p - 1)), R^1 \gamma_{\bar{\eta}} \underline{Cous}(X, \mathcal{F})^\bullet), \end{aligned}$$

where the second limit is taken over all closed subschemes $D \subset X$ of pure codimension one. \square

Remark 2.17. It seems that it is impossible to replace in the formulation of Proposition 2.16 the sheaf $R^l \gamma_Z \underline{Cous}(X, \mathcal{F})^\bullet$ by a more natural sheaf $R^l \gamma_Z \mathcal{F}$. At least the induction step in the above proof will not be valid, because in general there is no map $R^1 \gamma_{Z'/Z} (R^l \gamma_Z \mathcal{F}) \rightarrow R^{l+1} \gamma_{Z'} \mathcal{F}$ in notations from the proof of Proposition 2.16.

Example 2.18. Suppose that $l = p$; then we get the map $\nu_p = \nu_{0 \dots p}: \mathbf{A}((0 \dots p), \mathcal{F}) \rightarrow \bigoplus_{\eta \in X^{(p)}} H_\eta^p(X, \mathcal{F})$.

There is a morphism of complexes

$$\nu_X : \mathbf{A}(X, \mathcal{F})^\bullet \rightarrow \underline{Cous}(X, \mathcal{F})^\bullet$$

that is equal to the map $(-1)^{\frac{p(p+1)}{2}} \nu_p$ on the $(0 \dots p)$ -type components of the adelic complex and equals zero on all the other components of the adelic complex.

Also, for any two sheaves \mathcal{F} and \mathcal{G} on X and a point $\eta \in X^{(p)}$, we have the natural morphism

$$\begin{aligned} H_{\bar{\eta}}^p(X, \mathcal{F}) \otimes \mathbf{A}(X, \mathcal{G})^q &= \mathbf{A}((p), (i_{\bar{\eta}})_* R^p \gamma_{\bar{\eta}} \mathcal{F}) \otimes \mathbf{A}(X, \mathcal{G})^q \rightarrow \mathbf{A}(\bar{\eta}, R^p \gamma_{\bar{\eta}} (\mathcal{F} \otimes \mathcal{G}))^q \xrightarrow{\nu_{\bar{\eta}}} \\ &\xrightarrow{\nu_{\bar{\eta}}} \bigoplus_{\xi \in \bar{\eta}^{(q)}} H_\xi^q(\bar{\eta}, R^p \gamma_{\bar{\eta}} (\mathcal{F} \otimes \mathcal{G})) \subset \bigoplus_{\xi \in X^{(p+q)}} H_\xi^{p+q}(X, \mathcal{F} \otimes \mathcal{G}). \end{aligned}$$

It is easily checked that this defines a morphism of complexes

$$\mu : \text{Cous}(X, \mathcal{F})^\bullet \otimes \mathbf{A}(X, \mathcal{G})^\bullet \rightarrow \text{Cous}(X, \mathcal{F} \otimes \mathcal{G})^\bullet$$

given by the formula

$$\mu(f \otimes g)_\eta = (-1)^{\epsilon(p,q)} \sum_{\eta_0 \dots \eta_{q-1}} (\nu_{\eta_{q-1}\eta} \circ \dots \circ \nu_{\eta_0\eta_1})(f_{\eta_0} \cdot \theta(g)_{\eta_0 \dots \eta_{q-1}\eta})$$

for any $f \in \text{Cous}(X, \mathcal{F})^p$, $g \in \mathbf{A}(X, \mathcal{G})^q$, and $\eta \in X^{(p+q)}$, where the sum is taken over all flags $\eta_0 \dots \eta_{q-1}$ of type $(p, p+1, \dots, p+q-1)$ such that $\eta \in \overline{\eta}_{q-1}$ and $f_{\eta_0} \cdot \theta(g)_{\eta_0 \dots \eta_{q-1}\eta} \in H_\eta^p(X, \mathcal{F} \otimes \mathcal{G})$, and $\epsilon(p, q) = pq + \frac{q(q+1)}{2}$.

Remark 2.19. The analogous product is well defined if one replaces \mathcal{F} by a complex of abelian sheaves \mathcal{F}^\bullet .

Remark 2.20. A coherent version of the product between the Cousin and the adelic complex was considered in [27].

Example 2.21. Multiplication of the adelic complex on the right by $1 \in \mathbb{Z} = \mathbb{Z}(X)$ coincides with the morphism ν_X .

In particular, if \mathcal{A} is a sheaf of associative rings on X , then $\text{Cous}(X, \mathcal{A})^\bullet$ is a right DG-module over the DG-ring $\mathbf{A}(X, \mathcal{A})^\bullet$.

Remark 2.22. Evidently, we also have the morphism of complexes of sheaves $\underline{\nu}_X : \underline{\mathbf{A}}(X, \mathcal{F})^\bullet \rightarrow \underline{\text{Cous}}(X, \mathcal{F})^\bullet$. Suppose that the sheaf \mathcal{F} on X is *Cohen–Macaulay* in the sense of [12], i.e., that the composition $\mathcal{F} \rightarrow \underline{\mathbf{A}}(X, \mathcal{F})^\bullet \rightarrow \underline{\text{Cous}}(X, \mathcal{F})^\bullet$ is a quasiisomorphism; then for any $i \geq 0$, the cohomology group $H^i(X, \mathcal{F})$ is a direct summand in the group $H^i(\mathbf{A}(X, \mathcal{F})^\bullet)$. One may expect that the map $\mathcal{F} \rightarrow \underline{\mathbf{A}}(X, \mathcal{F})^\bullet$ is a quasiisomorphism for any Cohen–Macaulay sheaf \mathcal{F} . For the particular case of this statement see Theorem 3.34. In particular, if \mathcal{A} is a Cohen–Macaulay sheaf of associative rings, then the ring $\bigoplus_{i \geq 0} H^i(X, \mathcal{A})$ is a direct summand as a ring in the associative ring $\bigoplus_{i \geq 0} H^i(\mathbf{A}(X, \mathcal{A})^\bullet)$.

The next statement is needed for the sequel.

Lemma 2.23. *Suppose that X is a Noetherian catenary scheme such that all irreducible components of X have the same finite dimension d . Let \mathcal{F} be a Cohen–Macaulay sheaf on X (see [12]); then for any p , $0 \leq p \leq d$, the map $\nu_p : \bigoplus_{\eta_0 \dots \eta_p} \mathcal{F}_{\eta_0} \rightarrow \bigoplus_{\eta \in X^{(p)}} H_\eta^p(X, \mathcal{F})$ is surjective, where the first direct sum is taken over all flags of type $(0 \dots p)$ on X .*

Proof. The proof is by induction on p . For $p = 0$, there is nothing to prove. Suppose that $p > 0$. Consider a collection $\{f_\eta\} \in \bigoplus_{\eta \in X^{(p)}} H_\eta^p(X, \mathcal{F})$. Note that for any point

$\eta \in X^{(p)}$, the sheaf $j_\eta^* \mathcal{F}$ is Cohen–Macaulay on X_η , where $j_\eta : X_\eta = \text{Spec}(\mathcal{O}_{X,\eta}) \rightarrow X$ is the natural morphism. Hence for each point $\eta \in X^{(p)}$, there exists a collection $\{g_\xi\}_{(\eta)} \in \bigoplus_{\xi \in X_\eta^{(p-1)}} H_\xi^{p-1}(X_\eta, j_\eta^* \mathcal{F})$ such that $d_\eta \{g_\xi\}_{(\eta)} = f_\eta$, where d_η denotes the differential in the

Cousin complex on X_η . We may suppose that $\{g_\xi\}_{(\eta)} = 0$ for almost all $\eta \in X^{(p)}$. By the induction hypothesis, for each point $\eta \in X^{(p)}$, there exists a collection $\{g_{\xi_0 \dots \xi_{p-1}}\}_{(\eta)} \in \bigoplus_{\xi_0 \dots \xi_{p-1}} \mathcal{F}_{\xi_0}$ such that $\nu_{p-1}\{g_{\xi_0 \dots \xi_{p-1}}\}_{(\eta)} = \{g_\xi\}_{(\eta)}$, where the direct sum is taken over all flags of type $(0 \dots p-1)$ on X_η . Again, we may suppose that $\{g_{\xi_0 \dots \xi_{p-1}}\}_{(\eta)} = 0$ for almost all $\eta \in X^{(p)}$. Finally, we put $\{f_{\eta_0 \dots \eta_p}\} = \{g_{\eta_0 \dots \eta_{p-1}}\}_{(\eta_p)}$. \square

2.3 Projection formula

Let X, Y be Noetherian catenary irreducible schemes, $f : X \rightarrow Y$ be a morphism such that for any point $\eta \in X$, we have $\dim(\overline{\eta}) \geq \dim(\overline{f(\eta)})$. Under the above hypothesis, for any sheaf \mathcal{F} on X , there is a canonical morphism of complexes $Cous(X, \mathcal{F})^\bullet \rightarrow Cous(Y, Rf_*\mathcal{F}[d])^\bullet$, where $d = \dim(f) = \dim(X) - \dim(Y)$. The definition of this morphism uses inclusions of complexes $\Gamma_Z(X, C(\mathcal{F})^\bullet) \hookrightarrow \Gamma_{\overline{f(Z)}}(Y, f_*C(\mathcal{F})^\bullet)$ for any closed subset $Z \subset X$, where $C(\mathcal{F})^\bullet$ is a flasque resolution of \mathcal{F} on X . The morphism $Cous(X, \mathcal{F})^\bullet \rightarrow Cous(Y, Rf_*\mathcal{F}[d])^\bullet$ consists of homomorphisms of type $f_* : H_\eta^p(X, \mathcal{F}) \rightarrow H_{f(\eta)}^{p-d}(Y, Rf_*\mathcal{F}[d])$, where $\dim(\overline{\eta}) = \dim(\overline{f(\eta)})$.

The following adelic projection formula holds true.

Proposition 2.24. *Let $f : X \rightarrow Y$ be as above and let \mathcal{F}, \mathcal{G} be two sheaves on X ; then the following natural diagram commutes up to the sign $(-1)^{d \cdot \deg_{\mathbf{A}}}$, where $\deg_{\mathbf{A}}$ is the degree of the components in the adelic complex:*

$$\begin{array}{ccc} Cous(X, \mathcal{F})^\bullet \otimes \mathbf{A}(Y, f_*\mathcal{G})^\bullet & = & Cous(X, \mathcal{F})^\bullet \otimes \mathbf{A}(Y, f_*\mathcal{G})^\bullet \\ \downarrow & & \downarrow \\ Cous(X, \mathcal{F})^\bullet \otimes \mathbf{A}(X, \mathcal{G})^\bullet & & Cous(Y, Rf_*\mathcal{F}[d])^\bullet \otimes \mathbf{A}(Y, \mathcal{G})^\bullet \\ \downarrow & & \downarrow \\ Cous(X, \mathcal{F} \otimes \mathcal{G})^\bullet & \longrightarrow & Cous(Y, Rf_*(\mathcal{F} \otimes \mathcal{G})[d])^\bullet. \end{array}$$

Proof. For any point $\eta \in X^{(p)}$ such that $\dim(\overline{\eta}) = \dim(\overline{f(\eta)})$, the following diagram commutes

$$\begin{array}{ccc} H_\eta^p(X, \mathcal{F}) \otimes (f_*\mathcal{G})_{f(\eta)} & = & H_\eta^p(X, \mathcal{F}) \otimes (f_*\mathcal{G})_{f(\eta)} \\ \downarrow & & \downarrow \\ H_\eta^p(X, \mathcal{F}) \otimes \mathcal{G}_\eta & & H_{f(\eta)}^{p-d}(Y, Rf_*\mathcal{F}[d]) \otimes (f_*\mathcal{G})_{f(\eta)} \\ \downarrow & & \downarrow \\ H_\eta^p(X, \mathcal{F} \otimes \mathcal{G}) & \longrightarrow & H_{f(\eta)}^{p-d}(Y, Rf_*(\mathcal{F} \otimes \mathcal{G})[d]), \end{array}$$

where $f(\eta) \in Y^{(q)}$. Hence the proposition follows from Lemma 2.25 and the explicit formula for the product between the Cousin and the adelic complexes. \square

Lemma 2.25. *Let $f : X \rightarrow Y$ be as above with $\dim(X) = \dim(Y)$, \mathcal{H} be a sheaf on X , and $(\xi_0 \dots \xi_r)$ be a flag on Y such that $\xi_l \in Y^{(l)}$ for all $l, 0 \leq l \leq r$; then for any element $h \in \mathcal{H}_X$, we have*

$$f_* \left(\sum_{\eta_0 \dots \eta_r} \nu_{\eta_0 \dots \eta_r}(h) \right) = \nu_{\xi_0 \dots \xi_r}(f_*(h)),$$

where $f_* : H_\eta^*(X, \mathcal{H}) \rightarrow H_{f(\eta)}^{*-d}(Y, Rf_*\mathcal{H}[d])$ are the natural homomorphisms and the sum is taken over all flags $\eta_0 \dots \eta_r$ on X such that $f(\eta_0 \dots \eta_r) = (\xi_0 \dots \xi_r)$ and $\eta_l \in X^{(l)}$ for all $l, 0 \leq l \leq r$.

Proof. The proof is by induction on r . Note that there are only finitely many schematic points $\eta_1 \in X^{(1)}$ such that $f(\eta_1) = \xi_1$. Hence, if we replace X and Y by $\bar{\eta}_1$ and $\bar{\xi}_1$, respectively, we see that the induction step is equivalent to the case $r = 1$. On the other hand, when $r = 1$ the residue maps $\nu_{\eta_0\eta_1}$ and $\nu_{\xi_0\xi_1}$ correspond to the differentials in the Cousin complexes $Cous(X, \mathcal{H})^\bullet$ and $Cous(Y, Rf_*\mathcal{H}[d])^*$. Therefore Lemma 2.25 is equivalent to the fact that the homomorphisms f_* define a morphism of the corresponding Cousin complexes. \square

2.4 Sheaves with controllable support

Definition 2.26. We say that a sheaf \mathcal{F} on a scheme X has *controllable support* if for any point $\eta \in X$, there exists a closed subset $Z_\eta \subset X$ such that $\eta \notin Z_\eta$ and $\lim_{\rightarrow} (i_Z)_*\gamma_Z\mathcal{F} = (i_{Z_\eta})_*\gamma_{Z_\eta}\mathcal{F}$, where the limit is taken over all closed subsets $Z \subset X$ such that $\eta \notin Z$.

Remark 2.27.

- (i) If \mathcal{F} has controllable support, then for any open subset $U \subset X$ the sheaf \mathcal{F}_U has also controllable support (indeed, for any closed subset $Z \subset X$ we have $(i_Z)_*\gamma_Z(\mathcal{F}_U) = ((i_Z)_*\gamma_Z\mathcal{F})_U$;
- (ii) any subsheaf in a sheaf with controllable support has also controllable support.

Examples 2.28.

- 1) If X has finitely many irreducible components, then a constant sheaf on X has controllable support.
- 2) If \mathcal{F} is a coherent sheaf on a Noetherian scheme X , then \mathcal{F} has controllable support. Indeed, all sheaves $(i_Z)_*\gamma_Z\mathcal{F}$ in the definition are coherent subsheaves in the sheaf \mathcal{F} .
- 3) Any Cohen–Macaulay sheaf on a Noetherian scheme has controllable support.
- 4) Let X be an irreducible one-dimensional scheme with infinitely many closed points. Consider the sheaf $\bigoplus_{x \in X} (i_x)_*\mathbb{Z}$, where x ranges over all closed points in X and $i_x : \text{Spec}(k(x)) \hookrightarrow X$ is the closed embedding of a point. Then the sheaf $\bigoplus_{x \in X} (i_x)_*\mathbb{Z}$ does not have controllable support.

Claim 2.29. If a sheaf \mathcal{F} on a scheme X has controllable support, then for any point $\eta \in X$ there exists an open subset $U_\eta \subset X$ containing η such that for any open subsets $V \subset U \subset U_\eta$ containing η the natural morphism of sheaves $\mathcal{F}_U \rightarrow \mathcal{F}_V$ is injective.

Proof. We have $\text{Ker}\{\mathcal{F}_U \rightarrow \mathcal{F}_V\} = ((i_Z)_*\gamma_Z\mathcal{F})_U$, where $Z = X \setminus V$. We put $U_\eta = X \setminus Z_\eta$; since $Z_\eta \subset Z$ and $U \subset X \setminus Z_\eta$, we get $((i_Z)_*\gamma_Z\mathcal{F})_U = ((i_{Z_\eta})_*\gamma_{Z_\eta}\mathcal{F})_U = 0$. \square

Proposition 2.30. *Suppose that the sheaf \mathcal{F} on a scheme X has controllable support; then the map θ is injective for any subset $M \subset S(X)_p$:*

$$\theta : \mathbf{A}(M, \mathcal{F}) \hookrightarrow C(M, \mathcal{F}) = \prod_{(\eta_0 \dots \eta_p) \in M} \mathcal{F}_{\eta_0}.$$

Proof. The proof is by induction on p . For $p = 0$, there is nothing to prove. Suppose that $p > 0$; then, by the induction hypothesis, we have

$$\begin{aligned} \mathbf{A}(M, \mathcal{F}) &= \prod_{\eta \in P(X)} \varinjlim_{U_\eta} \mathbf{A}(\eta M, \mathcal{F}_{U_\eta}) \hookrightarrow \prod_{\eta \in P(X)} \varinjlim_{U_\eta} \prod_{(\eta_1 \dots \eta_p) \in {}_\eta M} (\mathcal{F}_{U_\eta})_{\eta_1} \hookrightarrow \\ &\hookrightarrow \prod_{\eta \in P(X)} \prod_{(\eta_1 \dots \eta_p) \in {}_\eta M} \varinjlim_{U_\eta} (\mathcal{F}_{U_\eta})_{\eta_1} = \prod_{(\eta_0 \dots \eta_p) \in M} \mathcal{F}_{\eta_0}, \end{aligned}$$

where the injectivity of the second map follows from Claim 2.29. \square

Thus when \mathcal{F} has controllable support, each adele $f \in \mathbf{A}(M, \mathcal{F})$ is uniquely determined by its components $f_{\eta_0 \dots \eta_p} \in \mathcal{F}_{\eta_0}$, where $(\eta_0 \dots \eta_p)$ runs over flags in M .

Remark 2.31. Let \mathcal{F} be a sheaf with controllable support on a scheme X ; then for any subset $M \subset S(X)_p$, there is a natural inclusion $\bigoplus_{(\eta_0 \dots \eta_p) \in M} \mathcal{F}_{\eta_0} \subset \mathbf{A}(M, \mathcal{F})$.

Examples 2.32. Let \mathcal{F} be a sheaf with controllable support on X . For an irreducible closed subscheme $Z \subset X$, by \mathcal{F}_Z denote the stalk of \mathcal{F} at the generic point of Z .

1) Suppose that $\dim X = 1$; then the adelic group $\mathbf{A}((01), \mathcal{F}) \subset \prod_{x \in X} \mathcal{F}_X$ consists of all collections $\{f_{Xx}\} \in \prod_{x \in X} \mathcal{F}_X$ such that $f_{Xx} \in \text{Im}(\mathcal{F}_x \rightarrow \mathcal{F}_X)$ for almost all $x \in X$. In particular, we get rational adeles on a curve if $\mathcal{F} = \mathcal{O}_X$ (see [24], where rational adeles are called *repartitions*).

2) Suppose that $\dim X = 2$. Let us describe explicitly the arising adelic groups. The adelic group $\mathbf{A}((01), \mathcal{F}) \subset \prod_{C \subset X} \mathcal{F}_X$ consists of all collection $\{f_{XC}\}$ such that $f_{XC} \in \text{Im}(\mathcal{F}_C \rightarrow \mathcal{F}_X)$ for almost all irreducible curves $C \in X$. The adelic group $\mathbf{A}((12), \mathcal{F}) \subset \prod_{x \in C} \mathcal{F}_C$ consists of all collections $\{f_{Cx}\}$ such that $f_{Cx} \in \text{Im}(\mathcal{F}_x \rightarrow \mathcal{F}_C)$ for almost all points $x \in C$ for a fixed C . The adelic group $\mathbf{A}((02), \mathcal{F}) \subset \prod_{x \subset X} \mathcal{F}_X$ consists of all collections $\{f_{Xx}\}$ such that there exists a divisor $D \subset X$ such that $f_{Xx} \in \text{Im}((\mathcal{F}_{X \setminus D})_x \rightarrow \mathcal{F}_X)$ for any closed point $x \in X$. The adelic group $\mathbf{A}((012), \mathcal{F}) \subset \prod_{x \in C \subset X} \mathcal{F}_X$ consists of all collections $\{f_{XCx}\}$ satisfying the following condition. There exists a divisor $D \subset X$ and for each irreducible curve $C \subset X$, there is a divisor D_C such that $D_C(C) = D(C)$ (see Definition 2.13), and $f_{XCx} \in \text{Im}((\mathcal{F}_{X \setminus D_C})_x \rightarrow \mathcal{F}_X)$ for all flags $x \in C \subset X$. This is analogous to the construction given in [19], p. 751.

2.5 1-pure sheaves

Definition 2.33. We say that a sheaf \mathcal{F} on a Noetherian scheme X is 1-pure if for any point $\eta \in X^{(i)}$, we have $H_\eta^0(X, \mathcal{F}) = 0$ if $i \geq 1$ and also $H_\eta^1(X, \mathcal{F}) = 0$ if $i \geq 2$.

Equivalently, a sheaf \mathcal{F} is 1-pure if the following complex of sheaves is exact:

$$0 \rightarrow \mathcal{F} \rightarrow \underline{Cous}(X, \mathcal{F})^0 \rightarrow \underline{Cous}(X, \mathcal{F})^1.$$

Remark 2.34. For any open subset $U \subset X$, any point $\xi \in P(X)$, and a 1-pure sheaf \mathcal{F} on X , there are exact sequences of sheaves

$$\begin{aligned} 0 \rightarrow \mathcal{F}_U \rightarrow \bigoplus_{\eta \in X^{(0)}} (i_{\bar{\eta}})_* \mathcal{F}_\eta &\rightarrow \bigoplus_{\eta \in D(X \setminus U)} (i_{\bar{\eta}})_* H_\eta^1(X, \mathcal{F}), \\ 0 \rightarrow [\mathcal{F}]_\xi &\rightarrow \bigoplus_{\eta \in X^{(0)}} (i_{\bar{\eta}})_* \mathcal{F}_\eta \rightarrow \bigoplus_{\eta \in D(\xi)} (i_{\bar{\eta}})_* H_\eta^1(X, \mathcal{F}), \end{aligned}$$

where $D(X \setminus U)$ is the set of codimension one points in X that belong to the complement $X \setminus U$ and $D(\xi)$ is the set of codimension one points η in X such that $\xi \in \bar{\eta}$. In particular, the sheaves \mathcal{F}_U and $[\mathcal{F}]_\xi$ are 1-pure for any 1-pure sheaf \mathcal{F} .

Remark 2.35. It is easily shown that for a sheaf \mathcal{F} on a Noetherian scheme X , the sheaf $\bigoplus_{\eta \in X^{(0)}} (i_{\bar{\eta}})_* \mathcal{F}_\eta$ has controllable support. Hence any 1-pure sheaf on a Noetherian scheme has controllable support.

For any flag $F = (\eta_0 \dots \eta_i)$ and a subset $M \subset S(X)_p$, we put ${}_F M = \eta_i(\eta_{i-1}(\dots \eta_0(M)))$.

Proposition 2.36. Let \mathcal{F} be a 1-pure sheaf on a Noetherian irreducible scheme X . Suppose that the subset $M \subset S(X)_p$ and a natural number l , $0 \leq l \leq p$ satisfy the following condition: either $p \leq l + 1$, or for any flag $F = (\eta_0 \dots \eta_{l+1}) \in S(X)_{p+1}$, we have ${}_F M = \eta_{l+1}(\delta_0^{p-l} \dots \delta_l^p(M))$ or ${}_F M = \emptyset$. Then the boundary maps $C(\delta_l^p(M), \mathcal{F}) \rightarrow C(M, \mathcal{F})$ and $\mathbf{A}(\delta_l^p(M), \mathcal{F}) \rightarrow \mathbf{A}(M, \mathcal{F})$, $0 \leq l \leq p$ (see Proposition 2.6(ii)) are injective and we have

$$\mathbf{A}(\delta_l^p(M), \mathcal{F}) = \mathbf{A}(M, \mathcal{F}) \cap C(\delta_l^p(M), \mathcal{F}),$$

where the intersection is taken inside the group $C(M, \mathcal{F})$.

Proof. Since X is irreducible, the sheaf \mathcal{F} is a subsheaf in a constant sheaf and for any points $\eta, \xi \in X$ and any open subset U such that $\xi \in \bar{\eta}$, $\eta \in U$, and U is connected, the natural morphisms $\mathcal{F}_\xi \rightarrow \mathcal{F}_\eta$ and $\mathcal{F}(U) \rightarrow \mathcal{F}_\eta$ are injective. Thus we get the injectivity of the boundary maps for the direct product groups and hence, by Proposition 2.30, we get the injectivity of the boundary maps for the adelic groups. In particular, the left hand side of the equality is contained in the right hand side.

To prove the backward inclusion we use induction on p and l . For $p = 1$ and $l = 0, 1$, we have $\mathbf{A}(\delta_l^1(M), \mathcal{F}) = C(\delta_l^1(M), \mathcal{F})$ and the assertion is clear.

Suppose that $p > 1, l = 0$. Denote by A the right hand side of the needed equality. For each $\eta \in P(X)$, consider the image A_η of A under the natural map $C(\delta_0^p(M), \mathcal{F}) \rightarrow$

$C(\eta M, \mathcal{F})$, see Proposition 2.6(i). The group A_η coincides with the projection of A on the η -part in the direct product $C(M, \mathcal{F}) = \prod_{\eta \in P(X)} C(\eta M, [\mathcal{F}]_\eta)$ and hence $A_\eta = \varinjlim_{U_\eta} \mathbf{A}(\eta M, \mathcal{F}_{U_\eta}) \cap C(\eta M, \mathcal{F})$, where the intersection is taken in the group $C(\eta M, [\mathcal{F}]_\eta)$. Therefore, by Lemma 2.40, $A_\eta = \mathbf{A}(\eta M, \mathcal{F})$. Further, note that $\delta_p^0(M) = \bigcup_{\eta \in P(X)} \eta M$ and for any point $\xi \in P(X)$ there exists $\eta \in P(X)$ such that $\xi(\eta M) = \xi(\delta_0^p(M))$. Therefore, by Lemma 2.41, $A = \mathbf{A}(\delta_0^p(M), \mathcal{F})$.

Suppose that $p > 1$, $l > 0$; then, by definition, the right hand side is equal to

$$\prod_{\eta \in P(X)} \varinjlim_{U_\eta} \mathbf{A}(\eta M, \mathcal{F}_{U_\eta}) \cap C(\delta_l^p(M), \mathcal{F}) = \prod_{\eta \in P(X)} \left(\varinjlim_{U_\eta} \mathbf{A}(\eta M, \mathcal{F}_{U_\eta}) \cap C(\eta \delta_l^p(M), [\mathcal{F}]_\eta) \right).$$

Since $l > 0$, for each point $\eta \in P(X)$, we have $\eta \delta_l^p(M) = \delta_{l-1}^p(\eta M)$. On the other hand, by Proposition 2.30, $\mathbf{A}(\eta M, \mathcal{F}_{U_\eta}) \subset C(\eta M, \mathcal{F}_{U_\eta})$ and the intersection corresponding to the point η is actually contained in the subgroup $\varinjlim_{U_\eta} C(\delta_{l-1}^p(\eta M), \mathcal{F}_{U_\eta}) \subset C(\delta_{l-1}^p(\eta M), [\mathcal{F}]_\eta)$.

Hence, by the inductive assumption for \mathcal{F}_{U_η} , ηM , $p - 1$, and $l - 1$, the intersection considered above is equal to

$$\prod_{\eta \in P(X)} \varinjlim_{U_\eta} (\mathbf{A}(\eta M, \mathcal{F}_{U_\eta}) \cap C(\delta_{l-1}^p(\eta M), \mathcal{F}_{U_\eta})) = \prod_{\eta \in P(X)} \varinjlim_{U_\eta} \mathbf{A}(\delta_{l-1}^p(\eta M), \mathcal{F}_{U_\eta}) = \mathbf{A}(\delta_l^p(M), \mathcal{F}).$$

□

Remark 2.37. It follows from the proof of Proposition 2.36 that for any subsheaf \mathcal{F} of a constant sheaf on a scheme X with finitely many irreducible components, for any $M \subset S(X)_p$, and for any l , $0 \leq l \leq p$, there is the inclusion

$$\mathbf{A}(\delta_l^p(M), \mathcal{F}) \subset \mathbf{A}(M, \mathcal{F}) \cap C(\delta_l^p(M), \mathcal{F}).$$

Remark 2.38. The condition that \mathcal{F} is 1-pure in Proposition 2.36 might be replaced by a weaker condition but actually we do not need such improvement: all sheaves that we consider further are 1-pure. The same is true about the condition on the set M .

Example 2.39. Suppose that X is irreducible and $\dim(X) = 2$. Let $D \subset X$ be a closed irreducible codimension one subset with infinitely many closed points. We put $G = \bigoplus_{y \in D} \mathbb{Z}$, where the sum is taken over all closed points y on D . Let \mathcal{F} be the kernel of the natural map $\underline{G} \rightarrow \bigoplus_{y \in D} (i_y)_* \mathbb{Z}$ of sheaves on X . Then \mathcal{F} is a subsheaf in a constant sheaf but is not 1-pure on X . Further, consider the adele $f \in C((12), \mathcal{F})$ defined by $f_{Cx} = \{1_x\} \in G$ if $C = D$ and $f_{Cx} = 0$ otherwise. For $C = D$, we have $f_{Cx} \in (\mathcal{F}_{X \setminus \{x\}})_x$ and there are two inclusions $(\mathcal{F}_{X \setminus \{x\}})_x \subset (\mathcal{F}_{X \setminus C})_x$ and $(\mathcal{F}_{X \setminus \{x\}})_x \subset \mathcal{F}_C$. Therefore, $f \in \mathbf{A}((012), \mathcal{F}) \cap C((12), \mathcal{F})$ but $f \notin \mathbf{A}((12), \mathcal{F})$.

Lemma 2.40. *Let \mathcal{F} be 1-pure sheaf on a Noetherian scheme X . Consider the sheaf $\mathcal{G} = \mathcal{F}_V$ for some open subset $V \subset X$. Then for any subset $M \subset S(X)_p$, we have*

$$\mathbf{A}(M, \mathcal{G}) \cap C(M, \mathcal{F}) = \mathbf{A}(M, \mathcal{F}),$$

where the intersection is taken in the group $C(M, \mathcal{G})$.

Proof. The proof is by induction on p . For $p = 0$, there is nothing to prove. Suppose that $p > 0$; then, by definition, the left hand side is equal to

$$\prod_{\eta \in P(X)} \left(\varinjlim_{U_\eta} \mathbf{A}(\eta M, \mathcal{G}_{U_\eta}) \cap C(\eta M, [\mathcal{F}]_\eta) \right).$$

Consider a point $\eta \in P(X)$ and an open subset $U \subset X$ containing η . Suppose that U is small enough so that all irreducible components of $U \setminus V$ contain η and the natural morphisms $\mathcal{F}_U \rightarrow [\mathcal{F}]_\eta$, $\mathcal{G}_U \rightarrow [\mathcal{G}]_\eta$ are injective. It follows from the explicit description given in Remark 2.34 that $\mathcal{G}_U \cap [\mathcal{F}]_\eta = \mathcal{F}_U$, where the intersection is taken inside the sheaf $[\mathcal{G}]_\eta$. Therefore $C(N, \mathcal{G}_U) \cap C(N, [\mathcal{F}]_\eta) = C(N, \mathcal{F}_U)$ for any subset $N \subset S(X)_q$.

Consequently the intersection $\varinjlim_{U_\eta} \mathbf{A}(\eta M, \mathcal{G}_{U_\eta}) \cap C(\eta M, [\mathcal{F}]_\eta)$ is actually contained in the subgroup $\varinjlim_{U_\eta} C(\eta M, \mathcal{F}_{U_\eta}) \subset C(\eta M, [\mathcal{F}]_\eta)$ and, by the inductive hypothesis, we get the needed statement. \square

Lemma 2.41. *Let \mathcal{F} be a sheaf with controllable support on a scheme X and let N be a subset in $S(X)_p$ such that $N = \bigcup_{\alpha} N_\alpha$, where $N_\alpha \subset S(X)_p$ for each α . Suppose that for any point $\eta \in P(X)$ there exists α such that $\eta(N_\alpha) = \eta N$. Let A be a subgroup in $C(N, \mathcal{F})$ such that for any α , the image of A under the natural map $C(N, \mathcal{F}) \rightarrow C(N_\alpha, \mathcal{F})$ is contained inside $\mathbf{A}(N_\alpha, \mathcal{F})$. Then $A \subset \mathbf{A}(N, \mathcal{F})$.*

Proof. For a point $\eta \in P(X)$ let α be such that $\eta(N_\alpha) = \eta N$. Then the projection of A to the group $C(\eta N, [\mathcal{F}]_\eta) = C(\eta(N_\alpha), [\mathcal{F}]_\eta)$ from the direct product $C(N, \mathcal{F}) = \prod_{\eta \in P(X)} C(\eta N, [\mathcal{F}]_\eta)$ is contained in the group $\varinjlim_{U_\eta} \mathbf{A}(\eta(N_\alpha), \mathcal{F}_{U_\eta}) = \varinjlim_{U_\eta} \mathbf{A}(\eta N, \mathcal{F}_{U_\eta})$. Thus we get that $A \subset \mathbf{A}(N, \mathcal{F})$. \square

Let X be a Noetherian scheme. Consider an increasing sequence of natural numbers $i_0 \leq \dots \leq i_p$. Recall that $(i_0 \dots i_p)$ denotes the set of all flags from $S(X)_p$ of type $(i_0 \dots i_p)$. For any l , $0 \leq l \leq p$, we have $\delta_l^p(i_0 \dots i_p) = (i_0 \dots \hat{i}_l \dots i_p)$. From this one deduces that the subset $(i_0 \dots i_p) \subset S(X)_p$ satisfies the condition from Proposition 2.36 for any l , $0 \leq l \leq p$. Let $(j_0 \dots j_q)$ be a subsequence in $(i_0 \dots i_p)$, $q \leq p$. There are canonical maps $\alpha : C((j_0 \dots j_q), \mathcal{F}) \rightarrow C((i_0 \dots i_p), \mathcal{F})$ and $\beta : \mathbf{A}((j_0 \dots j_q), \mathcal{F}) \rightarrow \mathbf{A}((i_0 \dots i_p), \mathcal{F})$ that are compositions of the corresponding boundary maps. By Proposition 2.36, we get the following.

Corollary 2.42. *Let \mathcal{F} be a 1-pure sheaf on a Noetherian irreducible scheme X . Then for any sequences $(i_0 \dots i_p)$ and $(j_0 \dots j_q)$ as above, the maps α and β are injective and we have*

$$\mathbf{A}((j_0 \dots j_q), \mathcal{F}) = \mathbf{A}((i_0 \dots i_p), \mathcal{F}) \cap C((j_0 \dots j_q), \mathcal{F}),$$

where the intersection is taken inside the group $C((i_0 \dots i_p), \mathcal{F})$.

In what follows we will always imply the inclusion β when comparing adelic groups with different indices given by type.

2.6 \mathbf{A}' -adelic groups

We introduce a new type of adelic groups. Let X be a Noetherian catenary scheme such that all irreducible components of X have the same finite dimension d and let \mathcal{F} be an abelian sheaf on X .

Consider a strictly increasing sequence of natural numbers (i_0, \dots, i_p) such that $i_p \leq d$. Recall that $C((i_0 \dots i_p), \mathcal{F}) = \prod_{\eta_0 \dots \eta_p} \mathcal{F}_{\eta_0}$, where the product is taken over all flags of type $(i_0 \dots i_p)$ on X . We put l to be the depth of $(i_0 \dots i_p)$ (see Proposition 2.16).

Definition 2.43. Let the subgroup $\mathbf{A}'((i_0 \dots i_p), \mathcal{F}) \subseteq C((i_0 \dots i_p), \mathcal{F})$ consist of all elements $f \in C((i_0 \dots i_p), \mathcal{F})$ such that for any number m , $0 \leq m \leq l$ and a flag $\eta_{m+1} \dots \eta_p$ of type $(i_{m+1} \dots i_p)$ on X , there are only finitely many flags $\eta_0 \dots \eta_m$ of type $(0 \dots m)$ on X such that the composition of the residue maps $(\nu_{\eta_{m-1}\eta_m} \circ \dots \circ \nu_{\eta_0\eta_1})(f_{\eta_0 \dots \eta_m \eta_{m+1} \dots \eta_p}) \in H_{\eta_m}^m(X, \mathcal{F})$ is not zero. These new adelic groups are called *\mathbf{A}' -adelic groups*, while the old adelic groups will be called *\mathbf{A} -adelic groups*.

When $l = -1$, we have $\mathbf{A}'((i_0 \dots i_p), \mathcal{F}) = C((i_0 \dots i_p), \mathcal{F})$. When $l \geq 0$, for any number m , $0 \leq m \leq l$, there is a map

$$\nu'_{0 \dots m} : \mathbf{A}'((i_0 \dots i_p), \mathcal{F}) \rightarrow \prod_{\eta_{m+1} \dots \eta_p} \left(\bigoplus_{\eta \in X^{(m)}} H_{\eta}^m(X, \mathcal{F}) \right),$$

where for each flag $\eta_{m+1} \dots \eta_p$ of type $(i_{m+1} \dots i_p)$ on X , the direct sum is taken over all points $\eta \in X^{(m)}$ such that $\eta_{m+1} \in \bar{\eta}$ (compare with Proposition 2.16).

Remark 2.44.

- (i) By Proposition 2.16, the image $\theta(\mathbf{A}((i_0 \dots i_p), \mathcal{F})) \subset C((i_0 \dots i_p), \mathcal{F})$ is contained in $\mathbf{A}'((i_0 \dots i_p), \mathcal{F})$.
- (ii) It is readily seen that the analogue of Corollary 2.7(i), (ii) with $M = (i_0 \dots i_p)$ holds for the \mathbf{A}' -adelic groups.
- (iii) By reciprocity law, for any j , $0 \leq j \leq p$, the boundary map

$$d_j^p : C((i_0 \dots \hat{i}_j \dots i_p), \mathcal{F}) \rightarrow C((i_0 \dots i_p), \mathcal{F})$$

induces the map of the corresponding \mathbf{A}' -groups; define the *\mathbf{A}' -adelic complex* $\mathbf{A}'(X, \mathcal{F})^\bullet$ by formula

$$\mathbf{A}'(X, \mathcal{F})^p = \prod_{0 \leq i_0 < \dots < i_p \leq d} \mathbf{A}'((i_0 \dots i_p), \mathcal{F})$$

and with the differential induced from the complex $C(X, \mathcal{F})_{red}^\bullet$; we have the morphism of complexes $\mathbf{A}'(X, \mathcal{F})^\bullet \xrightarrow{\nu'_X} \text{Cous}(X, \mathcal{F})^\bullet$ defined in the same way as for the \mathbf{A} -adelic complex; the composition $\mathbf{A}(X, \mathcal{F})^\bullet \rightarrow \mathbf{A}'(X, \mathcal{F})^\bullet \rightarrow \text{Cous}(X, \mathcal{F})^\bullet$ is equal to ν_X .

- (iv) Given two sheaves \mathcal{F}, \mathcal{G} on X , the morphism of complexes $C(X, \mathcal{F})_{red}^\bullet \otimes C(X, \mathcal{G})_{red}^\bullet \rightarrow C(X, \mathcal{F} \otimes \mathcal{G})_{red}^\bullet$ does *not* induce a morphism of complexes $\mathbf{A}'(X, \mathcal{F})^\bullet \otimes \mathbf{A}'(X, \mathcal{G})^\bullet \rightarrow \mathbf{A}'(X, \mathcal{F} \otimes \mathcal{G})^\bullet$; also, the analogue of Corollary 2.42 is not true for the \mathbf{A}' -adelic groups.
- (v) We may consider the sheafified version $\underline{\mathbf{A}}'(X, \mathcal{F})^\bullet$ of the \mathbf{A}' -adelic complex; there is a natural morphism of complexes $\mathcal{F} \rightarrow \underline{\mathbf{A}}'(X, \mathcal{F})^\bullet$.

Lemma 2.45. *For any natural numbers i_0, \dots, i_l, p such that $0 \leq i_0 < \dots < i_l < p \leq d$ and $(i_0 \dots i_l) \neq (0 \dots (p-1))$, we have*

$$\mathbf{A}'((i_0 \dots i_l p), \mathcal{F}) = \prod_{\eta \in X^{(p)}} \mathbf{A}'((i_0 \dots i_l \eta), \mathcal{F}),$$

where the index $(i_0 \dots i_l \eta)$ stands for the set of all flags $\eta_0 \dots \eta_l \eta_p$ on X of type $(i_0 \dots i_l p)$ such that $\eta_p = \eta$. Also, we have

$$\mathbf{A}'((0 \dots (p-1) p), \mathcal{F}) = \prod_{\eta \in X^{(p)}} \mathbf{A}'((0 \dots (p-1) \eta), \mathcal{F}),$$

where the restricted product means that we consider the set of all collections $\{f_\eta\} \in \prod_{\eta \in X^{(p)}} \mathbf{A}'((0 \dots (p-1) \eta), \mathcal{F})$ such that for almost all points $\eta \in X^{(p)}$, the \mathbf{A}' -adele $f_\eta \in \mathbf{A}'((0 \dots (p-1) \eta), \mathcal{F}) \subset \mathbf{A}'((0 \dots (p-1) p), \mathcal{F})$ belongs to $\text{Ker}(\nu_{0 \dots p})$.

Proof. This follows immediately from the definition of \mathbf{A}' -adelic groups combined with the \mathbf{A}' -analogue of Corollary 2.7(ii) (see Remark 2.44(ii)). \square

The lack of the multiplicative structure is the main disadvantage of the \mathbf{A}' -adelic complex (see Remark 2.44(iv)). Nevertheless, the main advantage of the \mathbf{A}' -adelic complex is the following statement.

Theorem 2.46. *Suppose that X is a Noetherian catenary scheme such that all irreducible components of X have the same finite dimension d and the sheaf \mathcal{F} on X is Cohen–Macaulay in the sense of [12]. Then the morphism $\underline{\nu}'_X : \underline{\mathbf{A}}'(X, \mathcal{F})^\bullet \rightarrow \underline{\text{Cous}}(X, \mathcal{F})^\bullet$ is a quasiisomorphism.*

Corollary 2.47. *Under the assumptions from Theorem 2.46, the natural morphism $\mathcal{F} \rightarrow \underline{\mathbf{A}}'(X, \mathcal{F})^\bullet$ is a quasiisomorphism.*

Proof of Theorem 2.46. It is enough to prove that the morphism $\nu'_U : \mathbf{A}'(U, \mathcal{F}|_U)^\bullet \rightarrow \text{Cous}(U, \mathcal{F}|_U)^\bullet$ is a quasiisomorphism for any open subset $U \subset X$. The sheaf $\mathcal{F}|_U$ is Cohen–Macaulay on U , hence we may suppose that $U = X$.

Using several intermediate complexes, we transform the \mathbf{A}' -adelic complex into the Cousin complex. For each number p , $0 \leq p \leq d$, consider the following complex:

$$\begin{aligned} C_p^\bullet : 0 \rightarrow \prod_{0 \leq i \leq p} \mathbf{A}'((i), \mathcal{F}) \rightarrow \dots \rightarrow \prod_{0 \leq i_0 < \dots < i_l \leq p} \mathbf{A}'((i_0 \dots i_l), \mathcal{F}) \rightarrow \dots \rightarrow \mathbf{A}'((0 \dots p), \mathcal{F}) \rightarrow \\ \rightarrow \bigoplus_{\eta \in X^{(p+1)}} H_\eta^{p+1}(X, \mathcal{F}) \rightarrow \dots \rightarrow \bigoplus_{\eta \in X^{(d)}} H_\eta^d(X, \mathcal{F}). \end{aligned}$$

The differential in the first part of this complex coincides with that in the \mathbf{A}' -adelic complex, the differential in the second part coincides with that in the Cousin complex, and the differential in the middle $\mathbf{A}'((0 \dots p), \mathcal{F}) \rightarrow \bigoplus_{\eta \in X^{(p+1)}} H_\eta^{p+1}(X, \mathcal{F})$ is equal to the composition of the boundary map $\mathbf{A}'((0 \dots p), \mathcal{F}) \rightarrow \mathbf{A}'((0 \dots p, (p+1)), \mathcal{F})$ with the map $\nu'_{0 \dots (p+1)}$. Thus $C_0^\bullet = \text{Cous}(X, \mathcal{F})^\bullet$ and $C_d^\bullet = \mathbf{A}'(X, \mathcal{F})^\bullet$.

For each p , $1 \leq p \leq d$, there is a natural morphism of complexes $\varphi_p : C_p^\bullet \rightarrow C_{p-1}^\bullet$ that is equal to the natural projection for the first part of the complex C_p^\bullet , is equal to $\nu'_{0 \dots p}$ for the p -th terms of C_p^\bullet , and is equal to the identity maps for the second part of the complex C_p^\bullet . We have $\varphi_1 \circ \dots \circ \varphi_d = \nu'_X$. We prove by induction on p that the morphism φ_p is actually a quasiisomorphism for all X and \mathcal{F} as above.

For $p = 0$, there is nothing to prove. Suppose that $1 \leq p \leq d = \dim(X)$. By Lemma 2.23, the morphism φ_p is surjective. So we need to show that the kernel of φ_p is an exact complex. By construction, $\text{Ker}(\varphi_p)^\bullet$ is equal to the complex

$$\begin{aligned} 0 \rightarrow \mathbf{A}'((p), \mathcal{F}) \rightarrow \prod_{0 \leq i < p} \mathbf{A}'((ip), \mathcal{F}) \rightarrow \dots \rightarrow \prod_{0 \leq i_0 < \dots < i_l < p} \mathbf{A}'((i_0 \dots i_l p), \mathcal{F}) \rightarrow \dots \\ \dots \rightarrow \prod_{0 \leq i < p} \mathbf{A}'((0 \dots \hat{i} \dots p), \mathcal{F}) \rightarrow \text{Ker}(\nu'_{0 \dots p}) \rightarrow 0. \end{aligned}$$

For each schematic point $\eta \in X^{(p)}$, consider the natural morphism $j_\eta : X_\eta = \text{Spec}(\mathcal{O}_{X, \eta}) \rightarrow X$. Note that $j_\eta^* \mathcal{F}$ is Cohen–Macaulay on X_η . Hence, by the induction hypothesis, the following complex is exact:

$$\begin{aligned} 0 \rightarrow \mathcal{F}_\eta \rightarrow \prod_{0 \leq i < p} \mathbf{A}'((i), j_\eta^* \mathcal{F}) \rightarrow \dots \rightarrow \prod_{0 \leq i_0 < \dots < i_l < p} \mathbf{A}'((i_0 \dots i_l), j_\eta^* \mathcal{F}) \rightarrow \dots \\ \dots \rightarrow \mathbf{A}'((0 \dots (p-1)), j_\eta^* \mathcal{F}) \rightarrow H_\eta^p(X, \mathcal{F}) \rightarrow 0. \end{aligned}$$

Now we take the product of these complexes over all points $\eta \in X^{(k)}$. We get the exact complex

$$B_p^\bullet : 0 \rightarrow \prod_{\eta \in X^{(p)}} \mathcal{F}_\eta \rightarrow \dots \rightarrow \prod_{\eta \in X^{(p)}} \left(\prod_{0 \leq i_0 < \dots < i_l < p} \mathbf{A}'((i_0 \dots i_l \eta), j_\eta^* \mathcal{F}) \right) \rightarrow \dots$$

$$\dots \rightarrow \prod_{\eta \in X^{(p)}} \mathbf{A}'((0 \dots (p-1)\eta), j_{\eta}^* \mathcal{F}) \xrightarrow{\nu'_{0 \dots p}} \bigoplus_{\eta \in X^{(p)}} H_{\eta}^p(X, \mathcal{F}) \rightarrow 0,$$

where the restricted product is taken in the same sense as in Lemma 2.45 and, as above, the index $(i_0 \dots i_l \eta_p)$ means that we consider the set of all flags $\eta_0 \dots \eta_l \eta_p$ on X of type $(i_0 \dots i_l p)$ with fixed η_p . Finally, by Lemma 2.45, we see that $\text{Ker}(\varphi_p)^{\bullet} = \tau_{\leq p}(B_p^{\bullet})$, where $\tau_{\leq p}$ is the canonical truncation of a complex, and thus the complex $\text{Ker}(\varphi_p)^{\bullet}$ is exact. \square

The following technical result is needed for the sequel. For any adele h and an increasing sequence of natural numbers $(j_0 \dots j_q)$, by $h_{j_0 \dots j_q}$ denote the component of h that has type $(j_0 \dots j_q)$.

Lemma 2.48. *Under the assumptions from Theorem 2.46, consider an increasing sequence $(0 \dots li_{l+1} \dots i_p)$ of depth l and an adele $g \in \mathbf{A}'((0 \dots li_{l+1} \dots i_p), \mathcal{F})$ such that $\nu'_{0 \dots (l+1)}(g) = 0$. Then there exists an adele $h \in \prod_{0 \leq i \leq l} \mathbf{A}'((0 \dots \hat{i} \dots li_{l+1} \dots i_p), \mathcal{F})$ such that $(dh)_{0 \dots li_{l+1} \dots i_p} = g$, where d is the differential in the \mathbf{A}' -adelic complex.*

Proof. We use notations from the proof of Theorem 2.46. Fix a flag $\eta_{l+1} \dots \eta_p$ of type $(i_{l+1} \dots i_p)$. We have $g_{0 \dots l \eta_{l+1} \dots \eta_p} \in \mathbf{A}'((0 \dots l), j_{\eta_{l+1}}^* \mathcal{F})$. By the condition of the lemma, $g_{0 \dots l \eta_{l+1} \dots \eta_p}$ is a degree l cocycle in the complex C_l^{\bullet} constructed for the local scheme $X_{\eta_{l+1}}$ and the sheaf $j_{\eta_{l+1}}^* \mathcal{F}$. It follows from the proof of Theorem 3.34 that the complex C_l^{\bullet} is exact for $X_{\eta_{l+1}}$ and $j_{\eta_{l+1}}^* \mathcal{F}$. Therefore there exists an adele $h_{\eta_{l+1} \dots \eta_p} \in \prod_{0 \leq i \leq l} \mathbf{A}'((0 \dots \hat{i} \dots l), j_{\eta_{l+1}}^* \mathcal{F})$ such that $d_{\eta_{l+1}}(h_{\eta_{l+1} \dots \eta_p})_{0 \dots l} = f_{0 \dots l \eta_{l+1} \dots \eta_p}$, where $d_{\eta_{l+1}}$ is the differential in the \mathbf{A}' -adelic complex on $X_{\eta_{l+1}}$. By Lemma 2.45, the collection

$$h = \{h_{\eta_{l+1} \dots \eta_p}\} \in \prod_{\eta_{l+1} \dots \eta_p} \prod_{0 \leq i \leq l} \mathbf{A}'((0 \dots \hat{i} \dots l), j_{\eta_{l+1}}^* \mathcal{F})$$

belongs to the \mathbf{A}' -adelic group $\prod_{0 \leq i \leq l} \mathbf{A}'((0 \dots \hat{i} \dots li_{l+1} \dots i_p), \mathcal{F})$ and h satisfied the needed condition. \square

3 Adeles for homology sheaves

We give some particular example of a class of sheaves on a smooth variety X over a field such that for any sheaf \mathcal{F} from this class the morphism of complexes of sheaves $\mathcal{F} \rightarrow \underline{\mathbf{A}}(X, \mathcal{F})^{\bullet}$ is a quasiisomorphism, i.e., the adelic complex is in fact a flasque resolution for the sheaf \mathcal{F} . This class of sheaves naturally arises from homology theories. In addition, these sheaves are Cohen–Macaulay in the sense of [12] (see Corollary 3.9(i)).

3.1 Homology theories

Let k be a field and \mathcal{V}_k be the category of varieties over k .

Definition 3.1. A *weak homology theory* over k is a presheaf on \mathcal{V}_k in the Zariski topology with value in the category of graded abelian groups

$$X \mapsto \bigoplus_{n \in \mathbb{Z}} F_n(X), \quad (j : U \hookrightarrow X) \mapsto (j^* : F_n(X) \rightarrow F_n(U))$$

such that for any closed embedding $f : X' \hookrightarrow X$, there is a functorial homomorphism of graded abelian groups $f_* : F_n(X') \rightarrow F_n(X)$ satisfying the following axioms:

(WH1): for any open embedding $j : U \hookrightarrow X$ and a closed embedding $f : X' \hookrightarrow X$, the following diagram commutes:

$$\begin{array}{ccc} F_n(U') & \xleftarrow{(j')^*} & F_n(X') \\ \downarrow f_* & & \downarrow f_* \\ F_n(U) & \xleftarrow{j^*} & F_n(X), \end{array}$$

where $j' : U' = f^{-1}(U) \hookrightarrow X'$ is an open embedding;

(WH2): for any closed embedding $i : Z \hookrightarrow X$ there exists a long exact localization sequence

$$\dots \rightarrow F_n(Z) \xrightarrow{i_*} F_n(X) \xrightarrow{j^*} F_n(X \setminus Z) \xrightarrow{\partial_{XZ}} F_{n-1}(Z) \rightarrow \dots,$$

where $j : X \setminus Z \hookrightarrow X$ is an open embedding; in addition, for any for any closed embedding $f : X' \hookrightarrow X$ and any pair of closed subsets $i : Z \hookrightarrow X$, $i' : Z' \hookrightarrow X'$ such that $f(Z') \subseteq Z$, the following diagram commutes:

$$\begin{array}{ccccccc} \dots \rightarrow & F_n(Z') & \xrightarrow{i'_*} & F_n(X') & \xrightarrow{(j')^*} & F_n(X' \setminus Z') & \xrightarrow{\partial_{X'Z'}} F_{n-1}(Z') \rightarrow \dots \\ & \downarrow f_* & & \downarrow f_* & & \downarrow f_* \alpha^* & \downarrow f_* \\ \dots \rightarrow & F_n(Z) & \xrightarrow{i_*} & F_n(X) & \xrightarrow{j^*} & F_n(X \setminus Z) & \xrightarrow{\partial_{XZ}} F_{n-1}(Z) \rightarrow \dots, \end{array}$$

where $\alpha : f^{-1}(X \setminus Z) \hookrightarrow X \setminus Z$ is an open embedding.

Note that this is a modified version of the notion of a twisted homology theory from [5].

Let F_* be a weak homology theory over a field k . For an irreducible variety X over k and $n \in \mathbb{Z}$, we put $F_n(k(X)) = \varinjlim_U F_n(U)$, where the limit is taken over all open non-empty subsets $U \subset X$. Evidently, this definition is correct, i.e., $F_n(k(X))$ depends only on the birational class of X .

The same reasoning as in [5], Proposition 3.7 shows that for any variety X over k , there is a homological type spectral sequence $E_{p,q}^1(X, F_*) = \bigoplus_{\eta \in X_{(p)}} F_{p+q}(k(\eta)) \Rightarrow F_{p+q}(X)$,

where $X_{(p)}$ is the set of all points η on X such that $\dim(\overline{\eta}) = p$. The corresponding ascending filtration on $F_n(X)$ is defined by $\text{Im}(\varinjlim_{Z \in X_{\leq p}} F_n(Z) \rightarrow F_n(X)) \subseteq F_n(X)$, where

$X_{\leq p}$ is the set of all closed subsets Z in X of dimension at most p . For $p, q \in \mathbb{Z}$, we put $Gers(X, F_*, q)_p = E_{p,q}^1(X, F_*)$. Thus $Gers(X, F_*, q)_\bullet$ is a homological type complex.

Let us say that a variety X is *equidimensional* if all irreducible components of X have the same dimension. For an equidimensional variety X of dimension d , we put $Gers(X, F_*, n)^p = E_{d-p, n-d}^1(X, F_*) = \bigoplus_{\eta \in X^{(p)}} F_{n-p}(k(\eta))$. The cohomological type complex

$Gers(X, F_*, n)^\bullet$ is called the *Gersten complex* associated to the weak homology theory F_* . Given a collection $\{f_\eta\} \in \bigoplus_{\eta \in X^{(p)}} F_{n-p}(k(\eta))$, the set of all schematic points $\eta \in X^{(p)}$

such that $f_\eta \neq 0$ is called the *support* of the collection $\{f_\eta\}$.

It is readily seen that for any $q \in \mathbb{Z}$ the functors $F(q)_n(X) = H_n(Gers(X, F_*, q)_\bullet)$, $n \in \mathbb{Z}, n \geq 0$ also form a weak homology theory. For an irreducible variety X of dimension d , we have $F(q)_d(k(X)) = F_{d+q}(k(X))$ and $F(q)_n(k(X)) = 0$ for $n \neq d$. Therefore, $Gers(X, F(q)_*, 0)_\bullet = Gers(X, F_*, q)_\bullet$ and $Gers(X, F(q)_*, m)_\bullet = 0$ if $m \neq 0$.

Definition 3.2. For a weak homology theory F_* , the *homology sheaves* $\mathcal{F}_n, n \in \mathbb{Z}$ are the sheaves on \mathcal{V}_k in the Zariski topology associated to the presheaves $F_n, n \in \mathbb{Z}$.

We put \mathcal{F}_n^X to be the restriction of the sheaf \mathcal{F}_n to the variety X if we need to distinguish sheaves on different spaces. Thus for an irreducible variety X , we have $(\mathcal{F}_n^X)_X = F_n(k(X))$. Note that for an open subset $i_U : U \hookrightarrow X$, we have $(i_U)^* \mathcal{F}_n^X = \mathcal{F}_n^U$. We also denote by \mathcal{F}_n^U the sheaf $(\mathcal{F}_n^X)_U = (i_U)_* \mathcal{F}_n^U$ on X .

It is readily seen that for any irreducible variety X of dimension d and any $q \in \mathbb{Z}$, the sheaf $\mathcal{F}(q)_d^X$ is 1-pure (see Section 2.5).

Remark 3.3. Suppose that X is an equidimensional variety of dimension d over k , η is a schematic point on X , and $D \subset X$ is a divisor; then any element $f \in (\mathcal{F}(n)_d^{X \setminus D})_\eta$, $n \in \mathbb{Z}$, is equal to the restriction of an element from $F_{n+d}(X_\eta \setminus (D \cup R))$, where R is a closed subset in X such that all irreducible components of R have codimension at least two in X . This follows directly from definitions and the localization sequence.

For an equidimensional variety X , we may also consider the sheafified Gersten complex $\underline{Gers}(X, F_*, n)^\bullet$, namely $\underline{Gers}(X, F_*, n)^p = \bigoplus_{\eta \in X^{(p)}} (i_{\bar{\eta}})_* F_{n-p}(k(\eta))$, where for each

point $\eta \in X^{(p)}$, we consider $F_{n-p}(k(\eta))$ as a constant sheaf on $\bar{\eta}$. There is a morphism of complexes of sheaves $\mathcal{F}_n^X \rightarrow \underline{Cous}(X, \mathcal{F}_n^X)^\bullet \rightarrow \underline{Gers}(X, F_*, n)^\bullet$, where \mathcal{F}_n^X is considered as a complex concentrated in the zero term. Also, we have $H^q(\underline{Gers}(X, F_*, n)^\bullet) = \mathcal{F}(n-d)_{d-q}^X$ and there is a natural morphism of sheaves $\mathcal{F}_n^X \rightarrow \mathcal{F}(n-d)_d^X$, where $d = \dim(X)$.

For any equidimensional closed subset $Z \subset X$ of codimension p in X , we have $\gamma_Z \underline{Gers}(X, F_*, n)^\bullet = \underline{Gers}(Z, F_*, n-p)^\bullet[-p]$. In particular, $R^q \gamma_Z \underline{Gers}(X, F_*, n) = \mathcal{F}(n-d)_{d-q}^Z$, where $d = \dim(X)$. Also, there are natural morphisms of sheaves $\mathcal{F}_{n-p}^Z \rightarrow R^p \gamma_Z \underline{Gers}(X, F_*, n) = \mathcal{F}(n-d)_{d-p}^Z$ and $R^p \gamma_Z \mathcal{F}_n^X \rightarrow R^p \gamma_Z \underline{Gers}(X, F_*, n)$. However, in general there is no natural morphism between the sheaves \mathcal{F}_{n-p}^Z and $R^p \gamma_Z \mathcal{F}_n^X$.

Definition 3.4. We say that a weak homology theory F_* over a field k is a *homology theory locally acyclic in fibrations* (l.a.f. homology theory) if the following two conditions are satisfied:

- (H): for any finite morphism $f : X' \rightarrow X$, there is a functorial homomorphism of graded abelian groups $f_* : F_n(X') \rightarrow F_n(X)$ extending the one for closed embeddings and such that the axioms (WH1) and (WH2) are satisfied with f being a finite morphism;
- (LAF): if for a closed embedding $i : Z \hookrightarrow X$ and a point $\eta \in Z$ there exists a morphism $\pi : X \rightarrow Z$ such that $\pi \circ i = \text{id}_Z$ and π smooth at η , then there exists an open subset $U \subset X$ containing η such that the composition $F_n(Z) \rightarrow F_n(X) \rightarrow F_n(U)$ is zero for any $n \in \mathbb{Z}$.

We say that a scheme Y is of *geometric type* over k if $Y = \bigcap_{\alpha} U_{\alpha}$, where $\{U_{\alpha}\}$ is a collection of open subsets in a variety X over k . For such Y , we put $F_n(Y) = \varinjlim_U F_n(U)$, where the limit is taken over all open subsets $U \subset X$ containing Y . It follows easily that $F_n(Y)$ is independent in X . If Y is an equidimensional scheme of geometric type over k , then we put $F_n(Y^{\geq p}) = \varinjlim_Z F_n(Z)$, where the limit is taken over all closed subsets $Z \subset Y$ such that all irreducible components of Z have codimension at least p in Y .

The proof of the next statement is the same as the proof of Theorem 5.11 in [22] or Theorem 4.2 in [5].

Proposition 3.5. *Let F_* be an l.a.f. homology theory over a field k ; then for any regular irreducible local scheme Y of geometric type over k and any $n \in \mathbb{Z}$, there is a natural short exact sequence*

$$0 \rightarrow F_n(Y^{\geq p}) \rightarrow \bigoplus_{\eta \in Y^{(p)}} F_n(k(\eta)) \rightarrow F_{n-1}(Y^{\geq (p+1)}) \rightarrow 0,$$

where the first map takes each element $\alpha \in F_n(Z)$ to the restrictions of α to the generic points of all components in Z that have codimension p in Y .

Remark 3.6. The differential in the Gersten complex for Y equals to the composition of two corresponding maps in the exact triples from Proposition 3.5 for n and for $n + 1$.

Corollary 3.7. *Under the notations from Proposition 3.5, any cocycle $\{f_{\eta}\} \in \bigoplus_{\eta \in Y^{(p)}} F_n(k(\eta))$ in the Gersten complex on Y is equal to the restriction of an element $\alpha \in F_n(Z)$ such that the set of all codimension p components of Z is equal to the support of $\{f_{\eta}\}$.*

Proof. By Proposition 3.5, any cocycle $\{f_{\eta}\} \in \bigoplus_{\eta \in Y^{(p)}} F_n(k(\eta))$ is defined by an element $\alpha \in F_n(Z)$ for a certain closed subset $Z \subset Y$ of codimension at least p . Let $Z_0 \subset Z$ be a codimension p irreducible component in Y that is not from the support of $\{f_{\eta}\}$. Note that $F_n(k(Z_0)) = \varinjlim F_n(U)$, where the limit is taken over all open subsets $U \subset Z_0$ that are also open in Z . From the localization sequence it follows that, in fact, α belongs to $F_n(Z')$, where the closed subset $Z' \subset Z$ has the same irreducible components as Z except for Z_0 and Z_0 is replaced by a proper closed subset. This concludes the proof. \square

Proposition 3.5 also implies the following important statement.

Proposition 3.8. *Let F_* be an l.a.f. homology theory over a field k ; then for any equidimensional regular variety X over k and for any $n \in \mathbb{Z}$, the morphism of complexes of sheaves $\mathcal{F}_n^X \rightarrow \underline{\text{Gers}}(X, F_*, n)^\bullet$ is a quasiisomorphism.*

Corollary 3.9. *Let F_* be an l.a.f. homology theory over a field k , X be an equidimensional regular variety X over k , and $n \in \mathbb{Z}$; then we have:*

- (i) *the sheaf \mathcal{F}_n^X on X is Cohen–Macaulay in the sense of [12] and the natural morphism of complexes of sheaves $\underline{\text{Cous}}(X, \mathcal{F}_n^X)^\bullet \rightarrow \underline{\text{Gers}}(X, F_*, n)^\bullet$ is an isomorphism; in particular, the sheaves \mathcal{F}_n^X are 1-pure on X .*
- (ii) *for any equidimensional closed subset $Z \subset X$ of codimension p , the natural morphism of sheaves $R^p\gamma_Z \mathcal{F}_n^X \rightarrow R^p\gamma_Z \underline{\text{Gers}}(X, F_*, n)^\bullet = \mathcal{F}(n-d)_{d-p}^Z$ is an isomorphism; in particular, there is a natural morphism $\mathcal{F}_{n-p}^Z \rightarrow R^p\gamma_Z \mathcal{F}_n^X$, which is an isomorphism at the generic points of Z ;*
- (iii) *for any $q \in \mathbb{Z}$, we have $\mathcal{F}(q)_d^X = \mathcal{F}_{d+q}^X$ and $\mathcal{F}(q)_m^X = 0$ if $m \neq 0$.*

Corollary 3.10. *Let F_* be an l.a.f. homology theory. Suppose X is an equidimensional variety that is regular outside of a codimension two closed subset; then for any point $\eta \in X$ and any $n \in \mathbb{Z}$, we have*

$$(\mathcal{F}(n)_d^X)_\eta = \bigcap_D F_{n+d}(\mathcal{O}_{X,D}) \subset F_{n+d}(k(X)),$$

where $d = \dim(X)$ and D runs over all irreducible divisors in X containing η .

Proof. This follows from the regularity of the discrete valuation ring $\mathcal{O}_{X,D}$ for any D and the exactness of the Gersten complex for $X_D = \text{Spec}(\mathcal{O}_{X,D})$. \square

Examples 3.11.

The following examples are homology theories locally acyclic in fibrations:

1) $F_n(X) = K'_n(X) = \pi_{n+1}(BQM(X))$ for $n \geq 0$ and $F_n(X) = 0$ for $n < 0$, where $\mathcal{M}(X)$ is the exact category of coherent sheaves on X (see [22], proof of Theorem 5.11).

2) $F_n(X) = H_n(X, i)$ for some $i \in \mathbb{Z}$, where (H^*, H_*) is a Poincaré duality theory with supports in the sense of Bloch–Ogus (see [5], Proposition 4.5).

3) $F_n(X) = A_n(X; M)$, where M is a cycle module over k in the sense of Rost (see [23], proof of Proposition 6.4). In this case we have $\text{Gers}(X, F_*, 0)_\bullet = C_\bullet(X; M)$ in notations from [23] and $\text{Gers}(X, F_*, m)_\bullet = 0$ if $m \neq 0$.

Remark 3.12. The sheaf $\mathcal{K}_n(\mathcal{O}_X)$ on a smooth variety X over k from both Examples 3.11, 1) and Examples 3.11, 3) for $M = \bigoplus_{n \geq 0} K_n$ (compare with Corollary 3.9, (iii)).

In the next three sections we develop some technique necessary for the proof of Lemma 3.37.

3.2 Strongly locally effaceable pairs

Let F_* be a homology theory locally acyclic in fibrations over a field k and X be an equidimensional variety. Consider equidimensional subvarieties $Z \subset X$ and $\tilde{Z} \subset X$ of codimensions p and $p-1$ in X , respectively, such that $Z \subset \tilde{Z}$.

Definition 3.13. Suppose that for each (not necessary closed) point $x \in Z$ and for any open subset $V \subset X$ containing x , there exists a smaller open subset $W \subset X$ containing x such that the natural map

$$F_n(V \cap Z) \rightarrow F_n(W \cap \tilde{Z})$$

is zero for all $n \in \mathbb{Z}$. In addition, suppose that for any $q \geq 0$, there exists an assignment $R \mapsto \Lambda(R)$, where R is an equidimensional subvariety of codimension q in Z , $\Lambda(R)$ is an equidimensional subvariety of codimension q in \tilde{Z} such that $R \subset \Lambda(R)$ and for any (not necessary closed) point $x \in Z$, for any open subset $V \subset X$ containing x , there exists a smaller open subset $W \subset X$ containing x such that the composition

$$F_n(V \cap (Z \setminus R)) \rightarrow F_n(V \cap (Z \setminus \Lambda(R))) \rightarrow F_n(W \cap (\tilde{Z} \setminus \Lambda(R)))$$

is zero for all $n \in \mathbb{Z}$ (in fact, this condition makes sense whenever $x \in R$). Then we say that the pair of subvarieties (Z, \tilde{Z}) is a *strongly locally effaceable pair*, or an *s.l.e. pair* (developing the terminology from [5]).

The assignment Λ in the definition of s.l.e. pairs is needed to establish a relation with the Gersten complex, as is stated in the next proposition.

Proposition 3.14. *Suppose that the field k is infinite and perfect. Let (Z, \tilde{Z}) be an s.l.e. pair of subvarieties on a smooth variety X over the field k such that Z has codimension p in X . Choose an arbitrary (not necessary closed) point $x \in Z$. Suppose that the collection $\{f_z\} \in \bigoplus_{z \in Z_x^{(0)}} F_n(k(z))$ is a cocycle in the Gersten complex on $X_x = \text{Spec}(\mathcal{O}_{X,x})$, i.e., suppose that $d_x(\{f_z\}) = 0$, where d_x is the differential in the complex $\text{Gers}(X_x, F_*, n+p)^\bullet$. Then there exists a collection $\{g_{\tilde{z}}\} \in \bigoplus_{\tilde{z} \in \tilde{Z}_x^{(0)}} F_{n+1}(k(\tilde{z}))$ such that $d_x(\{g_{\tilde{z}}\}) = \{f_x\}$.*

Proof. It follows from Corollary 3.7 that the collection $\{f_z\}$ is defined by an element $\alpha \in F_n((Z \cup S)_x)$ for a certain closed subset $S \subset X$ such that each irreducible component of S has codimension at least $p+1$ in X and is not contained in Z . We may assume that S is equidimensional of codimension $p+1$ in X . Hence the intersection $Z \cap S$ is contained in some equidimensional subvariety $R \subset Z$ of codimension $p+2$ in X . Furthermore, $\alpha \in F_n(V \cap (Z \cup S))$ for some open subset $V \subset X$ containing x .

By Corollary 3.18, there is an equidimensional subvariety \tilde{S} of codimension p in X such that the pair $(\Lambda(R) \cup S, \tilde{S})$ is strongly locally effaceable. Consider the following

commutative diagram, whose middle column is exact in the middle term:

$$\begin{array}{ccccc}
& F_n(\Lambda(R) \cup S) & \longrightarrow & F_n(\tilde{S}) & \\
& \downarrow & & \downarrow & \\
F_n(Z \cup S) & \longrightarrow & F_n(\tilde{Z} \cup S) & \longrightarrow & F_n(\tilde{Z} \cup \tilde{S}) \\
\downarrow & & \downarrow & & \\
F_n(Z \setminus R) & \longrightarrow & F_n(\tilde{Z} \setminus (\Lambda(R) \cup S)) & &
\end{array}$$

The map in the bottom row is the composition

$$F_n(Z \setminus R) \rightarrow F_n(\tilde{Z} \setminus \Lambda(R)) \rightarrow F_n(\tilde{Z} \setminus (\Lambda(R) \cup S)).$$

Since the pairs (Z, \tilde{Z}) and $(\Lambda(R) \cup S, \tilde{S})$ are s.l.e., for the point $x \in Z$ and the open subset $V \subset X$ considered above, there exists a smaller open subset $W \subset X$ containing x such that the map

$$F_n(V \cap (Z \cup S)) \rightarrow F_n(W \cap (\tilde{Z} \cup \tilde{S}))$$

is zero. Therefore α is a coboundary of an element $\beta \in F_{n+1}(W \cap ((\tilde{Z} \cup \tilde{S}) \setminus (Z \cup S)))$ in the localization exact sequence associated to the closed embedding $W \cap (Z \cup S) \hookrightarrow W \cap (\tilde{Z} \cup \tilde{S})$. In particular, β defines a collection $\{g_{\tilde{z}}\} \in \bigoplus_{\tilde{z} \in \tilde{Z}_x^{(0)}} F_{n+1}(k(\tilde{z}))$. Note that all codimension

$p - 1$ irreducible components of $\tilde{Z} \cup \tilde{S}$ are contained in \tilde{Z} , while all codimension p irreducible components of $Z \cup S$ are in Z (as before, codimensions are taken with respect to X). Therefore $d_x(\{g_{\tilde{z}}\}) = \{f_z\}$ and the proposition is proved. \square

3.3 Existence and addition of strongly locally effaceable pairs

Let F_* be an l.a.f. homology theory over a field k and X be a variety over k .

Definition 3.15. Let $f \geq 0$ be a natural number and (Z, \tilde{Z}) be an s.l.e. pair on X . Suppose that for each irreducible subvariety $C \subset X$ and an equidimensional subvariety $R \subset Z$ of codimension q in Z with $C \not\subseteq R$, we can choose an equidimensional subvariety $\Lambda_C(R) \subset \tilde{Z}$ of codimension q in \tilde{Z} such that $C \not\subseteq \Lambda_C(R)$, $R \subset \Lambda_C(R)$, and the following property holds true. For any f irreducible subvarieties $C_1, \dots, C_f \subset \tilde{Z}$, for any f equidimensional subvarieties $R_1, \dots, R_f \subset Z$ (maybe of different codimensions in Z) with $C_i \not\subseteq R_i$ for all $i, 1 \leq i \leq f$, for any schematic point $x \in Z$, and an open subset $V \subset X$ containing x , there exists a smaller open subset $W \subset X$ containing x such that the natural map

$$F_n(V \cap (Z \setminus (R_1 \cup \dots \cup R_f))) \rightarrow F_n(W \cap (\tilde{Z} \setminus (\Lambda_{C_1}(R_1) \cup \dots \cup \Lambda_{C_f}(R_f))))$$

is zero for all $n \in \mathbb{Z}$. Then we say that the pair of subvarieties (Z, \tilde{Z}) is *strongly locally effaceable with the freedom degree at least f* or is an *f -s.l.e. pair*. In particular, a *strongly locally effaceable pair with the freedom degree at least zero* is the same as a strongly locally effaceable pair.

Remark 3.16. If the pair (Z, \tilde{Z}) is f -s.l.e. and $Z' \subset Z$ is any closed equidimensional subset of the same dimension as Z , then the pair (Z', \tilde{Z}) is also f -s.l.e.

Here is the existence theorem for strongly locally effaceable pairs with a given freedom degree.

Theorem 3.17. *Suppose that the field k is infinite and perfect. Let X be an affine smooth variety over the field k . Consider an equidimensional subvariety Z of codimension $p \geq 2$ in X and a finite subset of closed points $T \subset X \setminus Z$. Then for any natural number $f \geq 0$, there exists a subvariety $\tilde{Z} \supset Z$ that does not contain any point from T and such that the pair (Z, \tilde{Z}) is strongly locally effaceable with the freedom degree at least f .*

Corollary 3.18. *Suppose that the field k is infinite and perfect. Let X be a smooth variety over the field k . Consider an equidimensional subvariety Z of codimension $p \geq 2$ in X and a closed subset $T \subset X$ such that no irreducible component of T is contained in Z and T has codimension at most $p - 1$ in X . Then there exists a subvariety $\tilde{Z} \supset Z$ that does not contain any irreducible component of T and such that the pair (Z, \tilde{Z}) is s.l.e.*

Proof. Consider a finite open affine covering $X = \cup_{\alpha} U_{\alpha}$. For each α and for each irreducible component of $T \cap U_{\alpha}$, choose a closed point on it outside of Z and thus get a finite subset $T'_{\alpha} \subset U_{\alpha} \setminus Z$. The application of Theorem 3.17 for the intersection of all data with U_{α} yields the existence of a closed subset $\tilde{Z}_{\alpha} \subset U_{\alpha}$ such that \tilde{Z}_{α} does not contain any point from T'_{α} and the pair $(Z \cap U_{\alpha}, \tilde{Z}_{\alpha})$ is s.l.e. We put $\tilde{Z} = \cup_{\alpha} \overline{\tilde{Z}_{\alpha}}$, where the bar denotes the closure in X . By the codimension assumption, \tilde{Z} does not contain any irreducible component of T . Also, the pair (Z, \tilde{Z}) is s.l.e., where $\Lambda(R)$ can be taken as the union over α of the closures of $\Lambda_{\alpha}(R \cap U_{\alpha})$ for an equidimensional subvariety $R \subset Z$. \square

Corollary 3.19. *Suppose that the field k is infinite and perfect. Let X be a smooth quasiprojective variety over the field k . Consider an equidimensional subvariety Z of codimension $p \geq 2$ in X and a closed subset $T \subset X$ such that no irreducible component of T is contained in Z . Then there exists a subvariety $\tilde{Z} \supset Z$ that does not contain any irreducible component of T and such that the pair (Z, \tilde{Z}) is s.l.e.*

Proof. For each irreducible component of T we choose a closed point on it outside of Z . Thus we get a finite set of closed points $T' \subset X \setminus Z$. Since X is quasiprojective, there exists a finite open affine covering $X = \cup_{\alpha} U_{\alpha}$ such that for each α we have $T' \subset U_{\alpha}$. To conclude the proof, we repeat the same argument as in the proof of Corollary 3.18. \square

Combining Proposition 3.14 and Corollary 3.18, we get the following statement, which could be considered as a uniform version of Gersten conjecture for smooth varieties and has interest in its own right.

Corollary 3.20. *Suppose that the field k is infinite and perfect. Let X be a smooth variety over the field k . Then for any equidimensional subvariety $Z \subset X$ of codimension p in X there exists an equidimensional subvariety $\tilde{Z} \supset Z$ of codimension $p - 1$ in X with the following property. Suppose we are given an arbitrary (not necessary closed)*

point $x \in Z$ and a collection $\{f_z\} \in \bigoplus_{z \in Z_x^{(0)}} F_n(k(z))$ such that $\{f_z\}$ is a cocycle in the local Gersten resolution at x , i.e., that $d_x(\{f_z\}) = 0$. Then there exists a collection $\{g_{\tilde{z}}\} \in \bigoplus_{\tilde{z} \in \tilde{Z}_x^{(0)}} F_{n+1}(k(\tilde{z}))$ such that $d_x(\{g_{\tilde{z}}\}) = \{f_z\}$.

Remark 3.21. Corollary 3.20 is stronger than Theorem 4.2 in [5] or Theorem 5.11 in [22]. Namely in [5] the analogous result was shown for a fixed subvariety Z , a fixed point $x \in Z$, and a fixed collection $\{f_z\}$ on Z_x . The proof in [5] does not seem to imply directly Corollary 3.20 and that is why we use some different geometrical method during the proof of Theorem 3.17.

Proof of Theorem 3.17. The proof is in two steps.

Step 1. During this step “a point” always means “a closed point”. Recall that $d = \dim X$. We say that a morphism $\pi : X \rightarrow \mathbb{A}^{d-1}$ *resolves* a point $x \in Z$ if π is smooth of relative dimension one at x , the restriction $\varphi = \pi|_Z$ is finite, $\varphi^{-1}(\varphi(x)) = \{x\}$, and $\pi(T) \cap \pi(Z) = \emptyset$. The following geometric result is a globalization of Quillen’s construction used in his proof of Gersten conjecture, see [22], Lemma 5.12 and [5], Claim on p.191.

Proposition 3.22. *Under the above assumptions, there exists a finite set Σ of morphisms $\pi : X \rightarrow \mathbb{A}^{d-1}$ such that for any f points $y_1, \dots, y_f \in X$ and any point $x \in Z$, there exists $\pi \in \Sigma$ such that π resolves x and $\pi(y_i) \notin \pi(Z \setminus \{y_i\})$ for all $i, 1 \leq i \leq f$.*

Proof. Using Claim 3.23, we prove by decreasing induction on e , $-1 \leq e \leq f$ that for any $e+1$ irreducible subsets Z_0, \dots, Z_e in Z there exist non-empty open subsets $U_0 \subset Z_0, \dots, U_e \subset Z_e$ and a finite set of morphisms Σ such that the statement of Proposition 3.22 is true for all collections of points $(x, y_1, \dots, y_f, y'_1, \dots, y'_f)$ satisfying $x \in U_0$, $y_i \in U_i$ for all $i, 1 \leq i \leq e$, $y_i \in Z$ for all $i, e+1 \leq i \leq f$, and $y'_i \in X \setminus Z$ for all $i, 0 \leq i \leq f$. Note that for $e = -1$ this immediately implies the needed statement of Proposition 3.22.

Suppose that $e = f$ and y'_1, \dots, y'_f are f arbitrary points in $X \setminus Z$. The application of Claim 3.23 for the points y'_1, \dots, y'_f and the closed subsets Z_0, \dots, Z_f in Z yields the existence of non-empty open subsets $U_0 \subset Z_0, \dots, U_f \subset Z_f$ and a morphism π satisfying the conditions from Claim 3.23. Since the conditions $y'_i \notin \pi^{-1}(\pi(Z))$, $1 \leq i \leq f$ are open, there is a finite open covering $\cup_\alpha V_\alpha$ of the direct product $(X \setminus Z)^{\times f}$ such that for all α , any collection (y'_1, \dots, y'_f) from the open subset V_α satisfies the conditions from Claim 3.23 with respect to some non-empty open subsets $U_0^\alpha \subset Z_0, \dots, U_f^\alpha \subset Z_f$ and a morphism π^α . Taking the finite intersections $U_i = \cap_\alpha U_i^\alpha$ for each $i, 0 \leq i \leq f$, we get the needed open subsets in Z_i , $0 \leq i \leq f$, while the needed finite set of morphisms is $\Sigma = \{\pi_\alpha\}$.

Now let us do the induction step from e to $e-1$. Choose any irreducible component C_1 of Z . By the inductive hypothesis, there exist non-empty open subsets $U_0^1 \subset Z_0, \dots, U_{e-1}^1 \subset Z_{e-1}, U_e^1 \subset C_1$ and a finite set of morphisms Σ_1 such that they satisfy the conditions stated above. We may assume that the subset $U_e^1 \subset C_1$ is also

open in Z . Let C_2 be one of the irreducible components of $Z \setminus U_e^1$. Again, by the inductive hypothesis, there exist other open subsets $U_0^2 \subset Z_0, \dots, U_{e-1}^2 \subset Z_{e-1}, U_e^2 \subset C_2$ and a finite set of morphisms Σ_2 such that they satisfy the conditions stated above. We repeat the same step until we come to the end of the obtained finite stratification of Z by open subsets U_e^j in C_j . Taking the finite intersections $U_i = \cap_j U_i^j$ for each $i, 0 \leq i \leq e-1$, we get the needed open subsets in $Z_i, 0 \leq i \leq e$, while the needed finite set of morphisms is equal to the finite union $\Sigma = \cup_j \Sigma_j$. \square

Claim 3.23. *For any f points $y'_1, \dots, y'_f \in X \setminus Z$ and $f+1$ closed subsets $Z_0, \dots, Z_f \subset Z$ there exist non-empty open subsets $U_i \subset Z_i, 0 \leq i \leq f$ and a morphism $\pi : X \rightarrow \mathbb{A}^{d-1}$ such that π resolves all point x from U_0 (with respect to Z), $\pi(y_i) \notin \pi(Z \setminus \{y_i\})$ for any point $y_i \in U_i, 1 \leq i \leq f$, and $\pi(y'_i) \notin \pi(Z)$ for all $i, 1 \leq i \leq f$.*

Proof. Let $\overline{X} \subset \mathbb{P}^N$ be a projective variety such that $X = \overline{X} \setminus H$, where $H \subset \mathbb{P}^N$ is a hyperplane. In what follows the bar denotes the projective closure in \mathbb{P}^N and the star denotes a join of two projective subvarieties in \mathbb{P}^N .

Without loss of generality we may assume that T contains the points y'_1, \dots, y'_f and that Z_i are irreducible for all $i, 0 \leq i \leq f$. Let x'_i be an arbitrary smooth point on Z_i for each $i, 0 \leq i \leq f$ (such x'_i exist because the field is perfect). We have the following dimension conditions: $\dim(H \cap \overline{Z}) \leq d-3$, $\dim(H \cap (T * \overline{Z})) \leq d-2$, $\dim(H \cap \overline{T_{x'_0} X}) = d-1$, and $\dim(H \cap \overline{T_{x'_i} Z_i}) \leq d-3$ for all $i, 0 \leq i \leq f$. Since the ground field is infinite, there exists a projective subspace $L' \subset H$ of codimension $d-2$ in H such that L' does not intersect with \overline{Z} , intersects $T * \overline{Z}$ in a finite set of points, intersects $\overline{T_{x'_0} X}$ in a line, and does not intersect with any $\overline{T_{x'_i} Z_i}$ for $0 \leq i \leq f$. Note that the projection $\pi_{L'}$ with the center at L' defines on \overline{Z} a finite morphism $\varphi_{L'}$. Put $Z'_i = \varphi_{L'}^{-1}(\varphi_{L'}(\overline{Z_i})) \subset \overline{Z}$ for $0 \leq i \leq f$. For each $i, 0 \leq i \leq f$, let x_i be an arbitrary point on $Z_i \subset Z'_i$ such that x_i is smooth on Z'_i , $\overline{T_{x_i} Z_i}$ does not intersect with L' , and $\overline{T_{x_0} X}$ intersects L' in a line.

We claim that the intersection $L' \cap (x_i * Z'_i)$ is a finite set of points for all $i, 0 \leq i \leq f$. Indeed, each join $x_i * Z'_i$ is the union two subsets. The first one is the tangent space to Z'_i at x_i and does not intersect with L' . The second one is the union of lines passing through x_i and other points from Z'_i . The intersection of this union of lines with L' corresponds to the fiber of x_i under the finite morphism $\varphi_{L'}$ and therefore is finite. Hence there exists a hyperplane $L \subset L'$ that does not intersect with the joins $T * \overline{Z}$ and $x_i * Z'_i$ for any $i, 0 \leq i \leq f$ and that intersects the tangent spaces $\overline{T_{x_0} X}$ in one point. Since the variety X is smooth, the projection π_L with the center at L is smooth at x_0 . Besides, the map π_L can not glue points from Z'_i with points from $\overline{Z} \setminus Z'_i$ for any $i, 0 \leq i \leq f$. Therefore the application of Lemma 3.24 with $Y = Z'_i$ yields that there exist non-empty open subsets $U_i \subset Z_i$ containing x_i such that π_L resolves all points x from U_0 and $\varphi_L^{-1}(\varphi_L(y_i)) = \{y_i\}$ for all points y_i from $U_i, 1 \leq i \leq f$, where $\varphi_L = \pi_L|_Z$. In addition, $\pi(T)$ does not intersect with $\pi(Z)$, and, in particular, $\pi(y'_i) \notin \pi(Z)$ for all $i, 1 \leq i \leq f$. \square

We have used the following fact from projective geometry.

Lemma 3.24. *Let $Y \subset \mathbb{P}^N$ be a projective variety, $x \in Y$ be a smooth point on Y . Suppose that a projective subspace $M \subset \mathbb{P}^N$ does not intersect with the join $x * Y$. Then*

there exists an open subset $W \subset Y$ containing x such that $\varphi^{-1}(\varphi(y)) = \{y\}$ for all $y \in W$, where φ is the restriction to Y of the projection π_M with the center at M .

Step 2. In notations from Proposition 3.22, consider the finite set Σ of morphisms $\pi : X \rightarrow \mathbb{A}^{d-1}$. Put $\tilde{Z} = \cup \pi^{-1}(\pi(Z))$, where the union is taken over all $\pi \in \Sigma$. By construction, \tilde{Z} does not contain any irreducible component of T .

Proposition 3.25. *The pair (Z, \tilde{Z}) is f -s.l.e.*

Proof. Essentially, we repeat the proof of Theorem 5.11 in [22] with some modifications.

First we note that after we choose a suitable closed point x' on $\bar{x} \subset Z$ we may suppose that the given open subset $V \ni x$ actually contains x' . Thus we may suppose x to be closed.

Choose $\pi \in \Sigma$ that resolves x and, as before, put $\varphi = \pi|_Z$. Following the construction of Quillen, consider the Cartesian square:

$$\begin{array}{ccc} Y & \xrightarrow{\varphi'} & X \\ \downarrow \pi' & & \downarrow \pi \\ Z & \xrightarrow{\varphi} & \mathbb{A}^{d-1}. \end{array}$$

Note that φ' is finite onto its image, $\varphi'(Y) = \pi^{-1}(\pi(Z))$ is a closed subset in \tilde{Z} , and $(\varphi')^{-1}(x)$ consists of one point, which we denote by z . Besides, the morphism π' is smooth at z and admits a canonical section $\sigma : Z \rightarrow Y$ (with $\sigma(x) = z$). Since the homology theory F_* is locally acyclic in fibrations, the composition $F_n(Z) \xrightarrow{\sigma_*} F_n(Y) \rightarrow F_n(Y')$ is zero for all $n \in \mathbb{Z}$ and for some suitable open subset $Y' \subset Y$ containing z . Hence the map $F_n(Z) \rightarrow F_n(\tilde{Z} \cap U)$ is also zero for all $n \in \mathbb{Z}$ and for some suitable open subset $U \subset X$ containing x such that $(\varphi')^{-1}(U) \subset Y'$ (such U exists, since φ' is finite and $(\varphi')^{-1}(x) = \{z\}$).

Take an arbitrary open subset $V \subset X$ containing x . Since $\varphi^{-1}(\varphi(x)) = \{x\}$ and φ is finite, there exists an open subset $D \subset \mathbb{A}^{d-1}$ such that $x \in \varphi^{-1}(D) \subset V$. Restricting the Cartesian diagram from \mathbb{A}^{d-1} to D , we get that the natural map $F_n(V \cap Z) \rightarrow F_n(W \cap \tilde{Z})$ is zero for all $n \in \mathbb{Z}$ and for some suitable open subset $W \subset V$ containing x .

Further, consider an irreducible subvariety $C \subset X$ and an equidimensional subvariety $R \subset Z$ of codimension q in Z such that $C \not\subseteq R$. We put

$$\Lambda_C(R) = \bigcup_{y \in C \setminus R} \left(\bigcup_{\pi \in \Sigma_y} \pi^{-1}(\pi(R)) \right),$$

where Σ_y is the set of all $\pi \in \Sigma$ such that $\pi(y) \notin \pi(Z \setminus \{y\})$. For instance, if C is not contained in \tilde{Z} , then $\Lambda_C(R) = \cup \pi^{-1}(\pi(R))$ where the union is taken over all $\pi \in \Sigma$. For irreducible subvarieties C_1, \dots, C_f in X and subvarieties R_1, \dots, R_f in Z satisfying the needed conditions, we choose closed points $y_i \in C_i \setminus R_i$. By construction, for any closed point $x \in Z$, there is a morphism $\pi \in \Sigma$ such that it resolves x and belongs to $\Sigma_{y_1} \cap \dots \cap \Sigma_{y_f}$. The same argument with the analogous Cartesian diagram as before leads to the needed result. \square

This completes the proof of Theorem 3.17. \square

Remark 3.26. In Theorem 3.17 one may also require that each irreducible component in \tilde{Z} contains some irreducible component in Z . This follows from the fact that for each irreducible component Z_0 in Z , the variety $\pi^{-1}(\pi(Z_0))$ is irreducible in an open neighborhood of a given point x , where $\pi : X \rightarrow \mathbb{A}^{d-1}$ is a morphism that resolves the point x and, in particular, is smooth at x .

The following proposition allows to add strongly locally effaceable pairs.

Proposition 3.27. *Suppose that the field k is infinite and perfect. Let X be an affine smooth variety over the field k . Consider two equidimensional subvarieties Z_1 and Z_2 of the same codimension $p \geq 2$ in X . Suppose that we are given a subvariety $\tilde{Z}_1 \supset Z_1$ such that the pair (Z_1, \tilde{Z}_1) is strongly locally effaceable with the freedom degree at least $f \geq 2$. Consider a closed subset $T \subset X$ such that all irreducible components of T have codimension at most $p-1$ in X , and an irreducible subvariety $K \subset X$ such that K is not contained in Z_2 . Then there exists a subvariety \tilde{Z}_2 such that no irreducible component of T and K is contained in \tilde{Z}_2 and the pair $(Z_1 \cup Z_2, \tilde{Z}_1 \cup \tilde{Z}_2)$ is strongly locally effaceable with the freedom degree at least $f-1$.*

Proof. If K is not contained in Z_1 , then the statement of the proposition follows directly from Theorem 3.17 after we choose a closed point on each irreducible component of T and K outside of $Z_1 \cup Z_2$ (in this case the freedom degree does not decrease). Otherwise we use the same construction as in the proof of Proposition 3.14.

Suppose that $K \subset Z_1$. Then there is a codimension one subvariety Z'_2 in Z_1 such that Z'_2 does not contain K and contains the intersection of Z_1 with each irreducible component of Z_2 that is not contained in Z_1 . Put $Z_3 = \Lambda_K(Z'_2) \subset \tilde{Z}_1$. By the codimension assumption, $Z_2 \cup Z_3$ does not contain any irreducible component of T . Choosing closed points on each irreducible component of T and on K outside of $Z_2 \cup Z_3$, we see that, by Theorem 3.17, there exists a subvariety $\tilde{Z}_2 \subset X$ such that the pair $(Z_2 \cup Z_3, \tilde{Z}_2)$ is $(f-1)$ -s.l.e. and \tilde{Z}_2 does not contain any irreducible component of T and K .

We claim that the pair $(Z_1 \cup Z_2, \tilde{Z}_1 \cup \tilde{Z}_2)$ is s.l.e. This is implied by the following commutative diagram, whose middle column is exact in the middle term:

$$\begin{array}{ccccc}
 & & F_n(Z_2 \cup Z_3) & \longrightarrow & F_n(\tilde{Z}_2) \\
 & & \downarrow & & \downarrow \\
 F_n(Z_1 \cup Z_2) & \longrightarrow & F_n(\tilde{Z}_1 \cup Z_2) & \longrightarrow & F_n(\tilde{Z}_1 \cup \tilde{Z}_2) \\
 \downarrow & & \downarrow & & \\
 F_n(Z_1 \setminus Z'_2) & \longrightarrow & F_n(\tilde{Z}_1 \setminus (Z_2 \cup Z_3)) & &
 \end{array}$$

The map in the bottom row is the composition

$$F_n(Z_1 \setminus Z'_2) \rightarrow F_n(\tilde{Z}_1 \setminus Z_3) \rightarrow F_n(\tilde{Z}_1 \setminus (Z_2 \cup Z_3)).$$

Since the pairs (Z_1, \tilde{Z}_1) and $(Z_2 \cup Z_3, \tilde{Z}_2)$ are s.l.e., for any point $x \in Z_1 \cup Z_2$ and any open subset $V \subset X$ containing x , there exists a smaller open subset $x \in W \subset X$ such that the map

$$F_n(V \cap (Z_1 \cup Z_2)) \rightarrow F_n(W \cap (\tilde{Z}_1 \cup \tilde{Z}_2))$$

is zero for all $n \in \mathbb{Z}$.

Now consider an irreducible subvariety $C \subset X$ and an equidimensional subvariety $R \subset Z_1 \cup Z_2$ of codimension q in $Z_1 \cup Z_2$ such that $C \not\subseteq R$. Let R' be the union of all irreducible components in R that are not contained in $Z_2' \cup Z_2$. Let $\Lambda'_C(R')$ be the union of all irreducible components in $\Lambda_C(R') \subset \tilde{Z}_1$ that are not contained in $Z_2 \cup Z_3$. Then there exists an equidimensional subvariety $R'' \subset Z_2 \cup Z_3$ of codimension q in $Z_2 \cup Z_3$ such that R'' contains the intersection $(\Lambda'_C(R') \cup R) \cap (Z_2 \cup Z_3)$ and does not contain C . Consider $\Lambda_C(R'') \subset \tilde{Z}_2$, where now Λ_C is taken with respect to the $(f-1)$ -s.l.e. pair $(Z_2 \cup Z_3, \tilde{Z}_2)$.

We claim that $(Z_1 \cup Z_2, \tilde{Z}_1 \cup \tilde{Z}_2)$ is an $(f-1)$ -s.l.e. pair with respect to the assignment $(R, C) \mapsto \Lambda_C(R) = \Lambda'_C(R') \cup \Lambda_C(R'')$. This is implied by the commutative diagram, which analogous to the previous one. This new diagram is the combination of two following diagrams:

$$\begin{array}{ccc}
F_n((Z_1 \cup Z_2) \setminus \{R_i\}) & \longrightarrow & F_n((\tilde{Z}_1 \cup Z_2) \setminus \{\Lambda'_{C_i}(R'_i) \cup R_i\}) \\
\downarrow & & \downarrow \\
F_n(Z_1 \setminus (Z_2' \cup \{R_i\})) & \longrightarrow & F_n(\tilde{Z}_1 \setminus (Z_2 \cup Z_3 \cup \{\Lambda'_{C_i}(R'_i)\})), \\
\\
F_n((Z_2 \cup Z_3) \setminus \{\Lambda'_{C_i}(R'_i) \cup R_i\}) & \longrightarrow & F_n(\tilde{Z}_2 \setminus \{\Lambda_{C_i}(R_i)\}) \\
\downarrow & & \downarrow \\
F_n((\tilde{Z}_1 \cup Z_2) \setminus \{\Lambda'_{C_i}(R'_i) \cup R_i\}) & \longrightarrow & F_n((\tilde{Z}_1 \cup \tilde{Z}_2) \setminus \{\Lambda_{C_i}(R_i)\}).
\end{array}$$

We glue these diagrams together using the following sequence, which is exact in the middle term:

$$\begin{aligned}
F_n((Z_2 \cup Z_3) \setminus \{\Lambda'_{C_i}(R'_i) \cup R_i\}) &\rightarrow F_n((\tilde{Z}_1 \cup Z_2) \setminus \{\Lambda'_{C_i}(R'_i) \cup R_i\}) \rightarrow \\
&\rightarrow F_n(\tilde{Z}_1 \setminus (Z_2 \cup Z_3 \cup \{\Lambda'_{C_i}(R'_i)\})).
\end{aligned}$$

Here $\{O_i\}$ means the union of objects O_i over all $i, 1 \leq i \leq f-1$ and the horizontal maps in the diagrams are the compositions of direct images under closed embedding and restrictions to open subsets. The new diagram is analyzed in the same way as the previous one. \square

Remark 3.28. For $p = 1$ by Quillen's result the only possible pair (Z, X) is s.l.e. However there is no analogue of Theorem 3.17 and Proposition 3.27 in this case with non-empty T and K .

3.4 Patching systems

Let k be a field and X be an equidimensional variety over k . Consider an equidimensional subvariety $Z \subset X$ of codimension p in X .

Definition 3.29. Suppose that the system of equidimensional subvarieties $\{Z_r^{1,2}\}$, $1 \leq r \leq p-1$ of codimension r in X , respectively, satisfies the following conditions:

- (i) the variety Z is contained in both varieties Z_{p-1}^1 and Z_{p-1}^2 , and the variety $Z_r^1 \cup Z_r^2$ is contained in both varieties Z_{r-1}^1 and Z_{r-1}^2 for all $r, 2 \leq r \leq p-1$;
- (ii) the pairs (Z, Z_{p-1}^1) , (Z, Z_{p-1}^2) , $(Z_r^1 \cup Z_r^2, Z_{r-1}^1)$, and $(Z_r^1 \cup Z_r^2, Z_{r-1}^2)$ are strongly locally effaceable with the freedom degree at least f for all $r, 2 \leq r \leq p-1$;
- (iii) the varieties Z_r^1 and Z_r^2 have no common irreducible components for all $r, 1 \leq r \leq p-1$.

Then we say that the system of subvarieties $\{Z_r^{1,2}\}$, $1 \leq r \leq p-1$ is a *patching system* for the subvariety Z with the freedom degree at least f .

Proposition 3.30. *Suppose that the field k is infinite and perfect; then for any integer $f \geq 0$ and any equidimensional subvariety Z of codimension p in the affine smooth variety X over the field k , there exists a patching system $\{Z_r^{1,2}\}$, $1 \leq r \leq p-1$ on X for the subvariety Z with the freedom degree at least f .*

Proof. We construct the needed system of subvarieties $\{Z_r^{1,2}\}$ by decreasing induction on r , $1 \leq r \leq p-1$.

Suppose that $r = p-1$. The application of Theorem 3.17 with empty T yields the existence of an equidimensional subvariety $Z_{p-1}^1 = \tilde{Z} \subset X$ of codimension $p-1$ in X such that (Z, Z_{p-1}^1) is an f -s.l.e. pair. Choosing a closed point on each irreducible component of Z_{p-1}^1 , we see that the application of Theorem 3.17 yields the existence of an equidimensional subvariety $Z_{p-1}^2 = \tilde{Z}$ of codimension $p-1$ in X such that Z_{p-1}^2 has no common irreducible components with Z_{p-1}^1 and (Z, Z_{p-1}^2) is an f -s.l.e. pair.

The induction step from $r+1$ to r , $1 \leq r < p-1$ is analogous to the case $r = p-1$ with Z replaced by $Z_{r+1}^1 \cup Z_{r+1}^2$. \square

Remark 3.31. Using Corollary 3.18 instead of Theorem 3.17 in the proof of Proposition 3.30, it is possible to show that for any equidimensional subvariety Z on a (not necessary affine) smooth variety X over an infinite perfect field there exists a patching system with the freedom degree at least zero.

Remark 3.32.

- (i) By Remark 3.26, in Proposition 3.30 one may also require that each irreducible component in Z_{p-1}^1 and Z_{p-1}^2 contains some irreducible component in Z and that for all r , $1 \leq r \leq p-2$, each irreducible component in Z_r^1 and Z_r^2 contains some irreducible component in $Z_{r+1}^1 \cup Z_{r+1}^2$.
- (ii) Let $W \subset X$ be an equidimensional subvariety such that W meets Z properly; if the patching system $\{Z_r^{1,2}\}$ satisfies the condition from (i), then W meets Z_r^i properly for all r , $1 \leq r \leq p-1$ and $i = 1, 2$. Combining this fact with Corollary 3.19, we get that under conditions of Proposition 3.30 one may also require that no irreducible component in $W \cap Z_r^1$ is contained in Z_r^2 for all r , $1 \leq r \leq p-1$.

Let us introduce the following notation. Suppose that $Z_i \subset X$, $p \leq i \leq q$ are equidimensional subvarieties of codimension i in an equidimensional variety X over k such that $Z_q \subset \dots \subset Z_p$. Consider a collection $f = \{f_\eta\} \in \bigoplus_{\eta \in X^{(p)}} F_n(k(\eta))$. We

put $\nu_{Z_p \dots Z_q}(f) = \sum_{z_p \dots z_q} (\nu_{k(z_{q-1})k(z_q)} \circ \dots \circ \nu_{k(z_p)k(z_{p-1})})(f_{z_p})$, where z_p, \dots, z_q range over all collections of generic points in Z_p, \dots, Z_q such that for any $i, p < i \leq q$, we have $z_i \in \bar{z}_{i-1}$. For $f \in F_n(k(X))$, let $\text{sing}(f)$ be the set of irreducible divisors D on X such that $\nu_{XD}(f) \neq 0$.

Here is the main property of patching systems.

Proposition 3.33. *Let $\{Z_r^{1,2}\}$, $1 \leq r \leq p-1$ be a patching system on X for the equidimensional subvariety $Z \subset X$ of codimension p in X with the freedom degree at least zero. Given a (not necessary closed) point $x \in Z$, suppose that a collection $g \in \bigoplus_{\eta \in X^{(p)}} F_{n-p}(k(\eta))$ is a cocycle in the local Gersten resolution $\text{Gers}(Y, F_*, n)^\bullet$ at x , where $Y = X_x$, and that the support of the collection g is contained in Z . Then there exists a collection $f \in \bigoplus_{\eta \in X^{(0)}} F_n(k(\eta))$ such that the subvariety $\text{sing}(\nu_{XZ_1^1 \dots Z_{r-1}^1}(f)) \subset Z_{r-1}^1$ is contained in $Z_r^1 \cup Z_r^2$ for all $r, 1 \leq r \leq p-1$, and $d_x \nu_{XZ_1^1 \dots Z_{p-1}^1}(f) = g$, where d_x is the differential in the local Gersten complex $\text{Gers}(Y, F_*, n)^\bullet$.*

Proof. The proof is by induction on $p \geq 1$. For $p = 1$, by Remark 3.28, there is nothing to prove.

Suppose that $p > 1$. Then, by Proposition 3.14, there exist two collections $g^1, g^2 \in \bigoplus_{\eta \in X^{(p-1)}} F_{n-p+1}(k(\eta))$ with the support on Z_{p-1}^1 and Z_{p-1}^2 , respectively, such that $d_x(g^i) = g$ for $i = 1, 2$. Therefore, $d_x(g^1 - g^2) = 0$ and, by the inductive assumption, there exists a collection $f \in \text{Gers}(X, F_*, n)^0$ such that $\text{sing}(\nu_{XZ_1^1 \dots Z_{r-1}^1}(f)) \subset Z_r^1 \cup Z_r^2$ for all $r, 1 \leq r \leq p-2$ and $d_x \nu_{XZ_1^1 \dots Z_{p-2}^1}(f) = g^1 - g^2$, where $Z_{p-1} = Z_{p-1}^1 \cup Z_{p-1}^2$. Such f satisfies the needed conditions with respect to the initial collection g . \square

3.5 Main theorem

Let F_* be a homology theory locally acyclic in fibrations over a field k .

Theorem 3.34. *Suppose that k is an infinite perfect field and that X is an irreducible smooth variety over k ; then for any $n \in \mathbb{Z}$, the morphism $\underline{\nu}_X : \underline{\mathbf{A}}(X, \mathcal{F}_n^X)^\bullet \rightarrow \underline{\text{Cous}}(X, \mathcal{F}_n^X)^\bullet = \underline{\text{Gers}}(X, F_*, n)^\bullet$ is a quasiisomorphism.*

Corollary 3.35. *Under the assumptions from Theorem 3.34, the natural morphism $\mathcal{F}_n^X \rightarrow \underline{\mathbf{A}}(X, \mathcal{F}_n^X)^\bullet$ is a quasiisomorphism; in particular, the cohomology groups $H^i(\underline{\mathbf{A}}(X, \mathcal{F}_n^X)^\bullet)$ are canonically isomorphic to the cohomology groups $H^i(X, \mathcal{F}_n^X)$.*

Remark 3.36. In Theorem 3.34 we make a strong restriction on the ground field to be infinite and perfect. In fact, the only one place where we use this is the geometric proof of Claim 3.23. It seems possible to prove the same result for smooth varieties over a

finite field, and, then, to reduce the case of regular varieties over an arbitrary field to that case by a standard argument of choosing a model. On the other hand, the author can prove Theorem 3.34 for $\dim X \leq 3$ over an arbitrary field, avoiding Claim 3.23.

Proof of Theorem 3.34. It is enough to prove that the morphism $\nu_U : \mathbf{A}(U, \mathcal{F}_n^U) \rightarrow \text{Cous}(U, \mathcal{F}_n^X)$ is a quasiisomorphism for any affine open subset $U \subset X$. We may put $X = U$. Since \mathcal{F}_n^X is a subsheaf in a constant sheaf, by Proposition 2.30, the complex $\mathbf{A}(X, \mathcal{F}_n^X)^\bullet$ is a subcomplex in the complex $\mathbf{A}'(X, \mathcal{F}_n^X)^\bullet$. By Remark 2.22 and Theorem 2.46, it is enough to show that for any p , $0 \leq p \leq d$, the natural homomorphism $H^p(\mathbf{A}(X, \mathcal{F}_n^X)^\bullet) \rightarrow H^p(\mathbf{A}'(X, \mathcal{F}_n^X)^\bullet)$ is injective.

For $p = 0$, there is nothing to prove. Suppose that $1 \leq p \leq d$ and consider an element $f \in \mathbf{A}(X, \mathcal{F}_n^X)^p$. Suppose that $f = d(g')$, where $g' \in \mathbf{A}'(X, \mathcal{F}_n^X)^{p-1}$. We want to show that there exists $g \in \mathbf{A}(X, \mathcal{F}_n^X)^{p-1}$ such that $d(g) = f$. We prove this by induction on the maximal depth l , $-1 \leq l \leq p-1$, of types for non-zero components of g' . Recall that for any adele h and an increasing sequence of natural numbers $(j_0 \dots j_q)$, by $h_{j_0 \dots j_q}$ we denote the component of h that has type $(j_0 \dots j_q)$.

Suppose that $l = -1$. Let $(i_0 \dots i_{p-1})$ be a sequence such that $g'_{i_0 \dots i_{p-1}} \neq 0$; then $i_0 > 0$ and we have $g'_{i_0 \dots i_{p-1}} = f_{0i_0 \dots i_{p-1}} \in \mathbf{A}((0i_0 \dots i_{p-1}), \mathcal{F}_n^X)$. By Corollary 2.42, $g'_{i_0 \dots i_{p-1}} \in \mathbf{A}((i_0 \dots i_{p-1}), \mathcal{F}_n^X)$ and thus $g' \in \mathbf{A}(X, \mathcal{F}_n^X)^{p-1}$.

Now suppose that $l \geq 0$. Let $(0 \dots li_{l+1} \dots i_{p-1})$ be a sequence such that $g'_{0 \dots li_{l+1} \dots i_{p-1}} \neq 0$. Suppose that $l < p-1$. Combining Proposition 2.16 and Corollary 3.9(ii), we get that the element $\nu_{0 \dots l+1}(f_{0 \dots l+1i_{l+1} \dots i_{p-1}})$ belong to the group

$$\begin{aligned} & \bigoplus_{\eta_{l+1} \in X^{(l+1)}} \mathbf{A}((0(i_{l+1} - l - 1) \dots (i_{p-1} - l - 1)), \mathcal{F}_{(n-d)_{d-l-1}}^{\bar{\eta}_{l+1}}) \subset \\ & \subset \prod_{\eta_{l+1} \dots \eta_{ip}} \left(\bigoplus_{\eta_{l+1} \in X^{(l+1)}} F_{n-l-1}(k(\eta_{l+1})) \right), \end{aligned}$$

where $d = \dim(X)$ and the product is taken over all flags $\eta_{i_{l+1}} \dots \eta_{i_p}$ of type $(i_{l+1} \dots i_p)$ and $\eta_{i_{l+1}} \in \bar{\eta}_{l+1}$. On the other hand, the reciprocity law implies that

$$\nu_{0 \dots l+1}(f_{0 \dots l+1i_{l+1} \dots i_{p-1}}) = \nu'_{0 \dots l+1}(-1)^{l+1} g'_{0 \dots li_{l+1} \dots i_{p-1}}.$$

By Lemma 3.37, there exists an element $g^0 \in \mathbf{A}((0 \dots li_{l+1} \dots i_{p-1}), \mathcal{F}_n^X)$ such that

$$\nu'_{0 \dots l+1}(g'_{0 \dots li_{l+1} \dots i_{p-1}} - g^0) = 0.$$

By Lemma 2.48, there exists an element $h' \in \prod_{0 \leq i \leq l} \mathbf{A}'((0 \dots \hat{i} \dots li_{l+1} \dots i_{p-1}), \mathcal{F}_n^X)$ such that $d(h')_{0 \dots li_{l+1} \dots i_{p-1}} = g'_{0 \dots li_{l+1} \dots i_{p-1}} - g^0$. Note that $d(g' - g^0 - d(h')) \in \mathbf{A}(X, \mathcal{F}_n^X)^p$ and the adele $g' - g^0 - d(h')$ has a strictly less nonzero depth l components than the adele g' . Therefore, by the inductive assumption, there exists an element $g^1 \in \mathbf{A}(X, \mathcal{F}_n^X)^{p-1}$ such that $d(g^1) = d(g' - g^0 - d(h'))$ and we put $g = g^0 + g^1$.

Now suppose that $l = p-1$; then, by Lemma 2.23, there exists an element $g^0 \in \mathbf{A}((0 \dots p-1), \mathcal{F}_n^X)$ such that $\nu'_{0 \dots p-1}(g' - g^0) = 0$. By Lemma 2.48, there exists an element

$h' \in \prod_{0 \leq i < p-1} \mathbf{A}'((0 \dots \hat{i} \dots p-1), \mathcal{F}_n^X)$ such that $d(h')_{0 \dots p-1} = (g' - g^0)_{0 \dots p-1}$. Note that $d(g' - g^0 - d(h')) \in \mathbf{A}(X, \mathcal{F}_n^X)^p$ and the adele $g' - g^0 - d(h')$ has no components of depth $p-1$. Therefore, by the inductive assumption, there exists an element $g^1 \in \mathbf{A}(X, \mathcal{F}_n^X)^{p-1}$ such that $d(g^1) = d(g' - g^0 - d(h'))$ and we put $g = g^0 + g^1$. \square

The essential part in the proof of Theorem 3.34 is the following approximation type lemma.

Lemma 3.37. *Under the assumptions from Theorem 3.34, consider an adele $f \in \mathbf{A}'((i_0 \dots i_p), \mathcal{F}_n^X)$ such that the depth of the sequence $(i_0 \dots i_p)$ is $l < p$ and*

$$\begin{aligned} \nu'_{0 \dots (l+1)}(f) &\in \bigoplus_{\eta_{l+1} \in X^{(l+1)}} \mathbf{A}((0(i_{l+1} - l - 1) \dots (i_p - l - 1)), \mathcal{F}(n - d)_{d-l-1}^{\bar{\eta}_{l+1}}) \subset \\ &\subset \prod_{\eta_{l+1} \dots \eta_{i_p}} \left(\bigoplus_{\eta_{l+1} \in X^{(l+1)}} F_{n-l-1}(k(\eta_{l+1})) \right), \end{aligned}$$

where $d = \dim(X)$ and the product is taken over all flags $\eta_{i_{l+1}} \dots \eta_{i_p}$ of type $(i_{l+1} \dots i_p)$ and with $\eta_{i_{l+1}} \in \bar{\eta}_{l+1}$. Then there exists an adele $g \in \mathbf{A}((i_0 \dots i_p), \mathcal{F}_n^X)$ of the same type as f such that $\nu_{0 \dots (l+1)}(g) = \nu'_{0 \dots (l+1)}(f)$.

Proof. During the proof η_s denotes a schematic point on X of codimension s in X (though sometimes points are considered on proper closed subvarieties in X). Further, d_{η_s} is the differential in the local Gersten resolution at η_s , i.e., in the complex $Gers(X_{\eta_s}, F_*, n)^\bullet$. For any two subvarieties C_1, C_2 in X , denote by $C_1 - C_2$ the union of all irreducible components of C_1 that are not contained in C_2 . Notice that for any two subvarieties C_1, C_2 in X , we have $(C_1 - C_2) \cup C_2 = C_1 \cup C_2$.

The proof is in two steps.

Step 1. Consider the collection of \mathbf{A} -adeles

$$\{h_{\eta_{l+1}(i_{l+1}-l-1) \dots (i_p-l-1)}\} = \nu'_{0 \dots (l+1)}(f) \in \bigoplus_{\eta_{l+1} \in X^{(l+1)}} \mathbf{A}((0(i_{l+1} - l - 1) \dots (i_p - l - 1)), \mathcal{F}_{n-l-1}^{\bar{\eta}_{l+1}}).$$

Let $Z_{l+1} = \cup \bar{\eta}_{l+1}$ be the union of the closures over the finite set of schematic points $\eta_{l+1} \in X^{(l+1)}$ such that $h_{\eta_{l+1}(i_{l+1}-l-1) \dots (i_p-l-1)}$ is a non-zero adele on $\bar{\eta}_{l+1}$. For each schematic point $\eta_{l+1} \in Z_{l+1}^{(0)}$ let $\{D_{\eta_{l+1}}, D_{\eta_{l+1}\eta_{i_{l+1}} \dots \eta_{i_k}}\}$, $l+1 \leq k < p$ be the system of divisors on $\bar{\eta}_{l+1}$ arising from the adelic condition for the \mathbf{A} -adele $h_{\eta_{l+1}(i_{l+1}-l-1) \dots (i_p-l-1)}$ on $\bar{\eta}_{l+1}$ (see Proposition 2.14).

Proposition 3.38. *For any flag $\eta_{i_{l+1}} \dots \eta_{i_k}$, $l+1 \leq k < p$ on Z_{l+1} , there exists an equidimensional subvariety $Z_{l+1; \eta_{i_{l+1}} \dots \eta_{i_k}} \subset X$ of codimension $l+1$ in X such that the system of subvarieties $\{Z_{l+1; \eta_{i_{l+1}} \dots \eta_{i_k}}\}$, $l+1 \leq k < p$ on X satisfies the following conditions:*

- (i) *for any point $\eta_{i_{l+1}}$ on Z_{l+1} , we have $Z_{l+1; \eta_{i_{l+1}}}(\eta_{i_{l+1}}) \subseteq Z_{l+1}(\eta_{i_{l+1}})$ and for any flag $\eta_{i_{l+1}} \dots \eta_{i_k}$, $l+1 < k < p$ on Z_{l+1} we have $Z_{l+1; \eta_{i_{l+1}} \dots \eta_{i_k}}(\eta_{i_k}) \subseteq Z_{l+1; \eta_{i_{l+1}} \dots \eta_{i_{k-1}}}(\eta_{i_{k-1}})$;*

(ii) for any flag $\eta_{l+1} \dots \eta_{i_k}$, $l+1 \leq k < p$ on Z_{l+1} , the subvariety $Z_{l+1;\eta_{l+1} \dots \eta_{i_k}} - Z_{l+1}$ contains the equidimensional subvariety

$$E_{\eta_{l+1} \dots \eta_{i_k}} = \bigcup_{\eta_{l+1} \in Z_{l+1}^{(0)}} (D_{\eta_{l+1}\eta_{l+1} \dots \eta_{i_k}} - D_{\eta_{l+1}})$$

of codimension $l+2$ in X and the pair $(E_{\eta_{l+1} \dots \eta_{i_k}}, Z_{l+1;\eta_{l+1} \dots \eta_{i_k}} - Z_{l+1})$ is strongly locally effaceable with the freedom degree at least $p-k$.

Proof. We construct the needed system of subvarieties $\{Z_{l+1;\eta_{l+1} \dots \eta_{i_k}}\}$ by induction on k , $l+1 \leq k < p$.

Suppose that $k = l+1$. Since for each point $\eta_{l+1} \in Z_{l+1}^{(0)}$ the system of divisors $\{D_{\eta_{l+1}}, D_{\eta_{l+1}\eta_{l+1} \dots \eta_{i_k}}\}$, $l+1 \leq k < p$ on $\bar{\eta}_{l+1}$ satisfies condition $(*)$ from Proposition 2.14, we get that for any point η_{l+1} on X , the defined above subvariety $E_{\eta_{l+1}}$ does not contain η_{l+1} . For each point η_{l+1} on Z_{l+1} , we choose closed points on each irreducible component of Z_{l+1} and on $\bar{\eta}_{l+1}$ outside of $E_{\eta_{l+1}}$ and thus get a finite set of closed points $T_{\eta_{l+1}} \subset X \setminus E_{\eta_{l+1}}$. The application of Theorem 3.17 with $f = p-l-1$, $Z = E_{\eta_{l+1}}$, and $T = T_{\eta_{l+1}}$ yields the existence of an equidimensional subvariety $z_{\eta_{l+1}} = \tilde{Z}$ of codimension $l+1$ in X such that $z_{\eta_{l+1}}$ does not contain $\bar{\eta}_{l+1}$, has no common irreducible components with Z_{l+1} , and $(E_{\eta_{l+1}}, z_{\eta_{l+1}})$ is an $(p-l-1)$ -s.l.e. pair. We put $Z_{l+1;\eta_{l+1}} = Z_{l+1} \cup z_{\eta_{l+1}}$.

Now we do the induction step from $k-1$ to k , $l+1 < k < p$. As before, by condition $(*)$ from Proposition 2.14, for any flag $\eta_{l+1} \dots \eta_{i_k}$ on Z_{l+1} , the subvariety

$$E_{\eta_{l+1} \dots \eta_{i_k}} - E_{\eta_{l+1} \dots \eta_{i_{k-1}}} = \bigcup_{\eta_{l+1} \in Z_{l+1}^{(0)}} (D_{\eta_{l+1}\eta_{l+1} \dots \eta_{i_k}} - (D_{\eta_{l+1}\eta_{l+1} \dots \eta_{i_{k-1}}} \cup D_{\eta_{l+1}}))$$

does not contain η_{i_k} . For each flag $\eta_{l+1} \dots \eta_{i_k}$, the application of Proposition 3.27 with $Z_1 = E_{\eta_{l+1} \dots \eta_{i_{k-1}}}$, $Z_2 = E_{\eta_{l+1} \dots \eta_{i_k}} - E_{\eta_{l+1} \dots \eta_{i_{k-1}}}$, $\tilde{Z}_1 = Z_{l+1;\eta_{l+1} \dots \eta_{i_{k-1}}} - Z_{l+1}$, $T = Z_{l+1}$, and $C = \bar{\eta}_{i_k}$ yields the existence of an equidimensional subvariety $z_{l+1;\eta_{l+1} \dots \eta_{i_k}} = \tilde{Z}_2$ of codimension $l+1$ in X such that $z_{l+1;\eta_{l+1} \dots \eta_{i_k}}$ does not contain $\bar{\eta}_{i_k}$, has no common irreducible components with Z_{l+1} , and

$$(E_{\eta_{l+1} \dots \eta_{i_{k-1}}} \cup E_{\eta_{l+1} \dots \eta_{i_k}}, (Z_{l+1;\eta_{l+1} \dots \eta_{i_{k-1}}} - Z_{l+1}) \cup z_{l+1;\eta_{l+1} \dots \eta_{i_k}})$$

is a $(p-k)$ -s.l.e. pair. We put $Z_{l+1;\eta_{l+1} \dots \eta_{i_k}} = Z_{l+1;\eta_{l+1} \dots \eta_{i_{k-1}}} \cup z_{l+1;\eta_{l+1} \dots \eta_{i_k}}$. By Remark 3.16,

$$(E_{\eta_{l+1} \dots \eta_{i_k}}, Z_{l+1;\eta_{l+1} \dots \eta_{i_k}} - Z_{l+1})$$

is a $(p-k)$ -s.l.e. pair and $Z_{l+1;\eta_{l+1} \dots \eta_{i_k}}(\eta_{i_k}) \subseteq Z_{l+1;\eta_{l+1} \dots \eta_{i_{k-1}}}(\eta_k)$. \square

Corollary 3.39. For any flag $\eta_{l+1} \dots \eta_{i_p}$ on Z_{l+1} , there exists a collection

$$\{g_{\eta_{l+1}\eta_{l+1} \dots \eta_{i_p}}\} \in \bigoplus_{\eta_{l+1} \in X_{\eta_{i_p}}^{(l+1)}} F_{n-l-1}(k(\eta_{l+1})),$$

satisfying the following conditions (note that the closure of the point η_{l+1} from the index of g may not contain $\eta_{i_{l+1}}$):

$$(i) \ d_{\eta_{i_p}}(\{g_{\eta_{l+1};\eta_{i_{l+1}}\dots\eta_{i_p}}\}) = 0;$$

$$(ii) \text{ if } \bar{\eta}_{l+1} \text{ contains } \eta_{i_{l+1}}, \text{ then } g_{\eta_{l+1};\eta_{i_{l+1}}\dots\eta_{i_p}} = h_{\eta_{l+1};\eta_{i_{l+1}}\dots\eta_{i_p}};$$

(iii) the support of the collection $\{g_{\eta_{l+1};\eta_{i_{l+1}}\dots\eta_{i_p}}\}$ is contained in the subvariety $Z_{l+1;\eta_{i_{l+1}}\dots\eta_{i_{p-1}}}$.

Proof. We use the notations from Proposition 3.38. Since the \mathbf{A}' -adele f has type $(0\dots l i_{l+1}\dots i_p)$, $i_{l+1} > l + 1$, by reciprocity law, we have $d_{\eta_{i_{l+1}}}(\{h_{\eta_{l+1};\eta_{i_{l+1}}\dots\eta_{i_p}}\}) = 0$ for any flag $\eta_{i_{l+1}}\dots\eta_{i_p}$ on Z_{l+1} , where $\{h_{\eta_{l+1};\eta_{i_{l+1}}\dots\eta_{i_p}}\}$ is considered as a collection from $Gers(X_{\eta_{i_{l+1}}}, F_*, n)^{l+1}$. Therefore the support of the collection $\{g_{\eta_{l+2};\eta_{i_{l+1}}\dots\eta_{i_p}}\} = d_{\eta_{i_p}}(\{h_{\eta_{l+1};\eta_{i_{l+1}}\dots\eta_{i_p}}\}) \in Gers(X_{\eta_{i_p}}, F_*, n)^{l+2}$ is contained in the subvariety $E_{\eta_{i_{l+1}}\dots\eta_{i_{p-1}}}$. Hence, by Propositions 3.14 and 3.38, there exists a collection $\{g'_{\eta_{l+1};\eta_{i_{l+1}}\dots\eta_{i_p}}\} \in Gers(X_{\eta_p}, F_*, n)^{l+1}$ with the support on $Z_{l+1;\eta_{i_{l+1}}\dots\eta_{i_{p-1}}} - Z_{l+1}$ such that $d_{\eta_{i_p}}(\{g'_{\eta_{l+1};\eta_{i_{l+1}}\dots\eta_{i_p}}\}) = g_{\eta_{l+2};\eta_{i_{l+1}}\dots\eta_{i_p}}$. Finally, we put $g_{\eta_{l+1};\eta_{i_{l+1}}\dots\eta_{i_p}} = \{h_{\eta_{l+1};\eta_{i_{l+1}}\dots\eta_{i_p}}\} - \{g'_{\eta_{l+1};\eta_{i_{l+1}}\dots\eta_{i_p}}\}$. \square

Step 2. By Proposition 3.30, there exists a patching system $\{Z_r^{1,2}\}$, $1 \leq r \leq l$ on X for the subvariety Z_{l+1} with the freedom degree at least $p - l$. We extend this patching system to patching systems for all subvarieties $Z_{l+1;\eta_{i_{l+1}}\dots\eta_{i_k}}$, $l + 1 \leq k < p$.

Proposition 3.40. *For each k , $l + 1 \leq k < p$, and for each flag $\eta_{i_{l+1}}\dots\eta_{i_k}$ on Z_{l+1} , there exists a patching system $Z_{r;\eta_{i_{l+1}}\dots\eta_{i_k}}^{1,2}$, $1 \leq r \leq l$ on X for the subvariety $Z_{l+1;\eta_{i_{l+1}}\dots\eta_{i_k}}$ with the freedom degree at least $p - k$, satisfying the following condition. Put $D = Z_1^1 \cup Z_1^2$ and $D_{\eta_{i_{l+1}}\dots\eta_{i_k}} = Z_{1;\eta_{i_{l+1}}\dots\eta_{i_k}}^1 \cup Z_{1;\eta_{i_{l+1}}\dots\eta_{i_k}}^2$ for any flag $\eta_{i_{l+1}}\dots\eta_{i_k}$, $l + 1 \leq k < p$ on Z_{l+1} . Then for any point $\eta_{i_{l+1}}$ on Z_{l+1} , we have $D_{\eta_{i_{l+1}}}(\eta_{i_{l+1}}) \subseteq D(\eta_{i_{l+1}})$ and for any flag $\eta_{i_{l+1}}\dots\eta_{i_k}$, $l + 1 < k < p$ on Z_{l+1} , we have $D_{\eta_{i_{l+1}}\dots\eta_{i_k}}(\eta_{i_k}) \subseteq D_{\eta_{i_{l+1}}\dots\eta_{i_{k-1}}}(\eta_{i_k})$.*

Proof. We use the notations from the proof of Proposition 3.38. The proof is by double induction on k and r , $l + 1 \leq k < p$, $1 \leq r \leq l$ (the induction on r is decreasing, as in the proof of Proposition 3.30).

Suppose that $k = l + 1$, $r = l$. For each point $\eta_{i_{l+1}}$ on Z_{l+1} , the application of Proposition 3.27 with $Z_1 = Z_{l+1}$, $Z_2 = z_{l+1;\eta_{i_{l+1}}}$, $\tilde{Z}_1 = Z_l^1$, $T = Z_l^2$, and $C = \bar{\eta}_{i_{l+1}}$ yields the existence of an equidimensional subvariety $z_{l;\eta_{i_{l+1}}}^1 = \tilde{Z}_2 \subset X$ of codimension l in X such that $z_{l;\eta_{i_{l+1}}}^1$ does not contain $\bar{\eta}_{i_{l+1}}$, has no common irreducible components with Z_l^2 , and $(Z_{l+1;\eta_{i_{l+1}}}, Z_l^1 \cup z_{l;\eta_{i_{l+1}}}^1)$ is a $(p - l - 1)$ -s.l.e. pair. We put $Z_{l;\eta_{i_{l+1}}}^1 = Z_l^1 \cup z_{l;\eta_{i_{l+1}}}^1$. Using Proposition 3.27 with $Z_1 = Z_{l+1}$, $Z_2 = z_{l+1;\eta_{i_{l+1}}}$, $\tilde{Z}_1 = Z_l^2$, $T = Z_{l+1;\eta_{i_{l+1}}}^1$, and $C = \bar{\eta}_{i_{l+1}}$, we get an equidimensional subvariety $z_{l;\eta_{i_{l+1}}}^2 = \tilde{Z}_2 \subset X$ of codimension l in X such that $z_{l;\eta_{i_{l+1}}}^2$ does not contain $\bar{\eta}_{i_{l+1}}$, has no common irreducible components with $Z_{l+1;\eta_{i_{l+1}}}^1$, and $(Z_{l+1;\eta_{i_{l+1}}}, Z_l^2 \cup z_{l;\eta_{i_{l+1}}}^2)$ is a $(p - l - 1)$ -s.l.e. pair. We put $Z_{l;\eta_{i_{l+1}}}^2 = Z_l^2 \cup z_{l;\eta_{i_{l+1}}}^2$.

The induction step from $r + 1$ to r for $k = l + 1$, $1 \leq r < l$ is analogous with Z_{l+1} replaced by $Z_{r+1}^1 \cup Z_{r+1}^2$, $z_{l+1;\eta_{l+1}}$ replaced by $z_{r+1;\eta_{l+1}}^1 \cup z_{r+1;\eta_{l+1}}^2$, and Z_l^j replaced by Z_r^j , $j = 1, 2$.

The reasoning for arbitrary k , $l + 1 < k < p$ is the same as for $k = l + 1$ with the subvarieties $\bar{\eta}_{l+1}$, $Z_{l+1;\eta_{l+1}}$, $z_{l+1;\eta_{l+1}}$ replaced by the subvarieties $\bar{\eta}_{i_k}$, $Z_{l+1;\eta_{l+1} \dots \eta_{i_k}}$, $z_{l+1;\eta_{l+1} \dots \eta_{i_k}}$, respectively, for each flag $\eta_{l+1} \dots \eta_{i_k}$ on Z_{l+1} and with the patching system $\{Z_r^{1,2}\}$, $1 \leq r \leq l$ replaced by the inductively defined patching system $\{Z_{r;\eta_{l+1} \dots \eta_{i_k-1}}^{1,2}\}$.

By construction, the system of divisors $\{D, D_{\eta_{l+1}, \dots, \eta_{i_k}}\}$, $l + 1 \leq k < p$ on X satisfies the needed condition. \square

Corollary 3.41. *For any flag $(\eta_{l+1} \dots \eta_{i_p})$ on Z_{l+1} , there exists a collection*

$$\{g_{\eta_0; \eta_{l+1} \dots \eta_{i_p}}\} \in \bigoplus_{\eta_0 \in X^{(0)}} F_n(k(\eta_0)),$$

such that $g_{\eta_0; \eta_{l+1} \dots \eta_{i_p}} \in (\mathcal{F}_n^{X \setminus D_{\eta_{l+1} \dots \eta_{i_p-1}}})_{\eta_{i_p}}$ and

$$d_{\eta_{l+1}} \nu_{X, Z_{1,F}^1 \dots Z_{l,F}^1}(\{g_{\eta_0; \eta_{l+1} \dots \eta_{i_p}}\}) = \{h_{\eta_{l+1} \eta_{l+1} \dots \eta_{i_p}}\}.$$

Proof. The corollary follows from the direct application of Proposition 3.33 for the collection $\{g_{\eta_{l+1}; \eta_{l+1} \dots \eta_{i_k}}\}$ from Corollary 3.39 and the patching system $\{Z_{r;\eta_{l+1} \dots \eta_{i_p-1}}^{1,2}\}$, $1 \leq r \leq l$ from Proposition 3.40. \square

Now we are ready to define the needed adele $g \in \mathbf{A}((i_0 \dots i_p), \mathcal{F}_n^X)$. Let $\eta_0 \dots \eta_l \eta_{l+1} \dots \eta_{i_p}$ be a flag of type $(i_0 \dots i_p)$ on X . Then we put $g_{\eta_0 \dots \eta_l \eta_{l+1} \dots \eta_{i_p}} = g_{\eta_0; (\eta_{l+1} \dots \eta_{i_p})}$ if $\eta_{l+1} \dots \eta_{i_p}$ is a flag on Z_{l+1} and $\eta_r \in (Z_{r,F}^1)^{(0)}$ for all $1 \leq r \leq l$. Otherwise we put $g_{\eta_0 \dots \eta_l \eta_{l+1} \dots \eta_{i_p}} = 0$.

For any flag $\eta_0 \dots \eta_k$ of type $(i_0 \dots i_k)$, $0 \leq k \leq l$ on X , we put $D_{\eta_0 \dots \eta_k} = D$. For any flag $\eta_0 \dots \eta_l \eta_{l+1} \dots \eta_{i_k}$ of type $(i_0 \dots i_k)$ on X , $l + 1 \leq k < p$, we put $D_{\eta_0 \dots \eta_l \eta_{l+1} \dots \eta_{i_k}} = D_{\eta_{l+1} \dots \eta_{i_k}}$ if $\eta_{l+1} \dots \eta_{i_k}$ is a flag on Z_{l+1} . Otherwise we put $D_{\eta_0 \dots \eta_l \eta_{l+1} \dots \eta_{i_k}} = \emptyset$.

By Proposition 3.40, the system of divisors $D_{\eta_0 \dots \eta_k}$, $0 \leq k < p$ on X satisfies condition $(*)$ from Proposition 2.14. By Corollary 3.41, the distribution g satisfies the adelic condition with respect to the system of divisors $D_{\eta_0 \dots \eta_k}$, $0 \leq k < p$ on X and we have $\nu_{0 \dots (l+1)}(g) = \nu'_{0 \dots (l+1)}(f)$.

This concludes the proof of Lemma 3.37. \square

3.6 Explicit cocycles

In this section we construct explicitly certain cocycles in the adelic complex corresponding to the given cocycles in the Gersten complex. Suppose k is an infinite perfect field, F_* is an l.a.f. homology theory over k , X is an irreducible smooth variety over the field k , and $Y \subset X$ is an equidimensional subvariety of codimension p in X . Consider a collection $\{f_y\} \in \bigoplus_{y \in Y^{(0)}} F_m(k(y))$ such that $\{f_\eta\}$ is a cocycle in the Gersten complex

$Gers(X, F_*, p + m)^\bullet$ on X . By Remark 3.31, there exists a patching system $\{Y_r^{1,2}\}$,

$1 \leq r \leq p-1$ on X for the subvariety Y with the freedom degree at least zero. We put $Y_p^1 = Y$.

Proposition 3.42. *There exists an adele $f = [\{f_y\}] \in \mathbf{A}(X, \mathcal{F}_{p+m}^X)^p$ such that f is a co-cycle in the adelic complex $\mathbf{A}(X, \mathcal{F}_{p+m}^X)^\bullet$ with $\nu_X(f) = \{f_y\}$, where $\nu_X : \mathbf{A}(X, \mathcal{F}_{p+m}^X)^\bullet \rightarrow \text{Gers}(X, F_*, p+m)^\bullet$ is the morphism from Theorem 3.34, and f satisfies the following conditions for any flag $\eta_{i_0} \dots \eta_{i_p}$ on X :*

- (i) $f_{\eta_{i_0} \dots \eta_{i_p}} = 0$ unless $\eta_{i_r} \in Y_r^1$ for all $r, 1 \leq r \leq p$;
- (ii) suppose that $\eta_{i_0} \notin Y_1^1$, $\eta_{i_r} \in Y_r^1$, $\eta_{i_r} \notin Y_r^2$ for all $r, 1 \leq r \leq p-1$ and $\eta_{i_p} \in Y$; then $f_{\eta_{i_0} \dots \eta_{i_p}} = \tilde{f}_{\eta_{i_p}} \in F_{p+m}(k(X))$, where $\tilde{f}_{\eta_{i_p}}$ depends only on η_{i_p} and satisfies the condition $d_{\eta_{i_p}} \nu_{XY_1^1 \dots Y_{p-1}^1}(\tilde{f}_{\eta_{i_p}}) = \{f_y\}_{\eta_{i_p}}$. Here $d_{\eta_{i_p}}$ is the differential in the Gersten resolution on $X_{\eta_{i_p}} = \text{Spec}(\mathcal{O}_{X, \eta_{i_p}})$, the notation $\nu_{XY_1^1 \dots Y_{p-1}^1}$ was introduced in Section 3.4, and the index η_{i_p} by a collection means that we consider the restriction of the collection to $X_{\eta_{i_p}}$;
- (iii) we have $\text{sing}(\nu_{XY_1^1 \dots Y_{p-1}^1}(f_{\eta_{i_0} \dots \eta_{i_p}})) \subset Y_r^1 \cup Y_r^2$ for all $r, 1 \leq r \leq l$, where l is the depth of the type $(i_0 \dots i_p)$.

Remark 3.43. Since $d(f) = 0$, we have $\nu_{0 \dots (l+1)}(d(f)) = 0$ for any integer $l, 0 \leq l \leq p-1$. Explicitly, for any flag $\eta_{i_1} \dots \eta_{i_{p-l+1}}$ of type $(i_1 \dots i_{p-l+1})$ on X with $i_1 > l$, we have

$$0 = d_{\eta_{i_1}} \nu_{0 \dots l} \left(\sum_{j=1}^{p-l+1} (-1)^{j+1} f_{01 \dots l \eta_{i_1} \dots \hat{\eta}_{i_j} \dots \eta_{i_{p-l+1}}} \right) \in \bigoplus_{\eta \in X_{\eta_{i_1}}^{(l+1)}} F_{p+m-l-1}(k(\eta)), \quad (**)$$

where for any flag Φ of type T on X , the index $(0 \dots l\Phi)$ means that we consider the set of all flags $\eta_0 \dots \eta_l \Phi$ of type $(0 \dots lT)$ on X with the fixed T -part Φ .

Proof of Proposition 3.42. We define the components of the adele f by decreasing induction on the depth $l, -1 \leq l \leq p$ of the type of a component. Moreover, we enlarge the induction hypothesis by condition (**).

Let $\eta_0 \dots \eta_{p-1} \eta_i$ be a flag on X with the type depth $l \geq p-1$, i.e., $i \geq p$. If $\eta_i \notin Y$, then we put $f_{\eta_0 \dots \eta_{p-1} \eta_i} = 0$. Suppose that $\eta_i \in Y$. Then, by Proposition 3.33, there exists an element $\tilde{f}_{\eta_i} \in F_{p+m}(k(X))$ such that $\text{sing}(\nu_{XY_1^1 \dots Y_{p-1}^1}(\tilde{f}_{\eta_i})) \subset Y_r^1 \cup Y_r^2$ for all $r, 1 \leq r \leq p-1$ and

$$d_{\eta_i} \nu_{XY_1^1 \dots Y_{p-1}^1}(\tilde{f}_{\eta_i}) = \{f_y\}_{\eta_i} \in \bigoplus_{\eta \in X_{\eta_i}^{(p)}} F_m(k(\eta)).$$

We put $f_{\eta_0 \dots \eta_{p-1} \eta_i} = (-1)^{\frac{p(p+1)}{2}} \tilde{f}_{\eta_i}$ if η_r is a generic point of some irreducible component of Y_r^1 for all $r, 1 \leq r \leq p-1$. Otherwise, we put $f_{\eta_0 \dots \eta_{p-1} \eta_i} = 0$. It is readily seen that conditions (i), (ii) are satisfied for the defined above $(0 \dots p-1, i)$ -type component of f , and also condition (**) holds for $l = p-1$ and any flag $\eta_{i_1} \eta_{i_2}$ of type $(i_1 i_2)$ on X such that $i_1 > p-1$.

Now we do the induction step from $l+1$ to l , $0 \leq l \leq p-2$. Let $\eta_0 \dots \eta_l \eta_{i_1} \dots \eta_{i_{p-l}}$ be a flag of type $(0 \dots l i_1 \dots i_{p-l})$ on X , $i_1 > l+1$. If $\eta_{i_1} \notin Y_{l+1}^1$, then we put $f_{\eta_0 \dots \eta_l \eta_{i_1} \dots \eta_{i_{p-l}}} = 0$. Suppose that $\eta_{i_1} \in Y_{l+1}^1$. Then, by the inductive assumption and by Proposition 3.33 applied to the patching system $\{Y_r^{1,2}\}$, $1 \leq r \leq l$ on X for the subvariety Y_{l+1}^1 and the point η_{i_1} , there exists an element $\tilde{f}_{\eta_{i_1} \dots \eta_{i_{p-l}}} \in F_{p+m}(k(X))$ such that $\text{sing}(\nu_{XY_1^1 \dots Y_l^1}(\tilde{f}_{\eta_{i_1} \dots \eta_{i_{p-l}}})) \subset Y_r^1 \cup Y_r^2$ for all r , $1 \leq r \leq l$ and we have

$$d_{\eta_{i_1}} \nu_{XY_1^1 \dots Y_l^1}(\tilde{f}_{\eta_{i_1} \dots \eta_{i_{p-l}}}) = \nu_{l+1} \left(\sum_{j=1}^{p-l} (-1)^{j+1} f_{0 \dots l+1 \eta_{i_1} \dots \hat{\eta}_{i_j} \dots \eta_{i_{p-l}}} \right)_{\eta_{i_1}} \in \bigoplus_{\eta \in X_{\eta_{i_1}}^{(l+1)}} F_{p+m-l-1}(k(\eta)).$$

We put $f_{\eta_0 \dots \eta_l \eta_{i_1} \dots \eta_{i_{p-l}}} = \tilde{f}_{\eta_{i_1} \dots \eta_{i_{p-l}}}$ if η_r is a generic point of some irreducible component of Y_r^1 for all r , $1 \leq r \leq l$. Otherwise, we put $f_{\eta_0 \dots \eta_l \eta_{i_1} \dots \eta_{i_{p-l}}} = 0$.

In the above notation suppose that $\eta_{i_r} \notin Y_{l+r}^1$ for some r , $1 < r \leq p-l$. Since for any j , $1 \leq j < r$, we have $\eta_{i_r} \notin Y_{l+r}^1$ and for any j , $r \leq j \leq p-l$, we have $\eta_{i_{r-1}} \notin Y_{l+r}^1$, by the induction hypothesis, we get that $f_{0 \dots l+1 \eta_{i_1} \dots \hat{\eta}_{i_j} \dots \eta_{i_{p-l}}} = 0$ for any j , $1 \leq j \leq p-l$. Therefore we may put $\tilde{f}_{\eta_{i_1} \dots \eta_{i_{p-l}}} = 0$ and hence the $(0 \dots l i_1 \dots i_{p-l})$ -type component of f satisfies condition (i).

Further, suppose that $\eta_{i_r} \in Y_{l+r}^1$, $\eta_{i_r} \notin Y_{l+r}^2$ for all r , $1 \leq r \leq p-1$ and $\eta_{i_p} \in Y$; then $\eta_{i_1} \notin Y_{l+2}^1$ and, by the inductive hypothesis, $f_{0 \dots l+1 \eta_{i_1} \dots \hat{\eta}_{i_j} \dots \eta_{i_{p-l}}} = 0$ for all j , $1 < j \leq p-l$ and $\tilde{f}_{0 \dots l+1 \eta_{i_1} \dots \eta_{i_{p-l}}} = \tilde{f}_{\eta_{i_{p-l}}}$, where $\tilde{f}_{\eta_{i_{p-l}}} \in F_{p+m}(k(X))$ satisfies condition (ii). Therefore we have the condition

$$d_{\eta_{i_1}} \nu_{XY_1^1 \dots Y_l^1}(\tilde{f}_{\eta_{i_1} \dots \eta_{i_{p-l}}}) = \nu_{XY_1^1 \dots Y_{l+1}^1}(\tilde{f}_{\eta_{i_{p-l}}}).$$

By the inductive hypothesis, we have $\text{sing}(\nu_{XY_1^1 \dots Y_l^1}(\tilde{f}_{\eta_{i_{p-l}}})) \subset Y_{l+1}^1 \cup Y_{l+1}^2$. Therefore we may put $\tilde{f}_{\eta_{i_1} \dots \eta_{i_{p-l}}} = \tilde{f}_{\eta_{i_{p-l}}}$ and hence the $(0 \dots l i_1 \dots i_{p-l})$ -type component of f satisfies condition (ii).

As above, it is a trivial check that condition $(**)$ holds for $l-1$ and any flag $\eta_{i_1} \dots \eta_{i_{p-l-1}}$ of type $(i_1 \dots i_{p-l-1})$ on X such that $i_1 > l-1$.

Finally, we put $f_{\eta_{i_0} \dots \eta_{i_p}} = \sum_{j=0}^p (-1)^j f_{0 \eta_{i_0} \dots \hat{\eta}_{i_j} \dots \eta_{i_p}}$ for any flag $\eta_{i_0} \dots \eta_{i_p}$ of type $(i_0 \dots i_p)$ on X with $i_0 > 0$. By the induction hypothesis, we have $f_{\eta_{i_0} \dots \eta_{i_p}} \in (\mathcal{F}_{p+m}^X)_{\eta_{i_0}}$. The same reasoning as above shows that conditions (i) and (ii) hold for the $(i_0 \dots i_p)$ -type component of f .

Since $\text{sing}(f_{\eta_{i_0} \dots \eta_{i_p}}) \subset Y_1^1 \cup Y_1^2$ for any flag $\eta_{i_0} \dots \eta_{i_p}$ on X , we see that the distribution f satisfies the adelic condition with respect to the constant system of divisors $Y_1^1 \cup Y_1^2$. Thus we have defined the needed cocycle $f \in \mathbf{A}(X, \mathcal{F}_{p+m}^X)^p$. \square

The following claim is necessary for the proof of Theorem 4.22.

Claim 3.44. *Under the above assumptions, consider a schematic point $\eta \in Y$ and suppose that the cocycle $\{f_y\}_\eta \in \text{Gers}(X_\eta, F_*, p+m)^p$ in the local Gersten resolution on X_η*

is the restriction of an element $\alpha \in F_m(Y_\eta)$. Then the element $\tilde{f}_\eta \in F_{p+m}(k(X))$ in condition (ii) from Proposition 3.42 may be chosen such that the collection $d_\eta \nu_{XY_1^1 \dots Y_{r-1}^1}(\tilde{f}_\eta)$ is the restriction of an element $\alpha_r \in F_{p+m-r}((Y_r^1 \cup Y_r^2)_\eta)$ for all $r, 1 \leq r \leq p-1$. Moreover, for all $r, 1 \leq r < p-1$, the restriction of α_r to $F_{p+m-r}((Y_r^1)_\eta \setminus Y_r^2)$ is equal to the restriction of an element from $F_{p+m-r}((Y_r^1)_\eta \setminus (Y_{r+1}^1 \cup Y_{r+1}^2))$ and the restriction of α_{p-1} to $F_{p+1}((Y_{p-1}^1)_\eta \setminus Y_{p-1}^2)$ is equal to the restriction of an element from $F_{p+1}((Y_{p-1}^1)_\eta \setminus Y)$. Finally, \tilde{f}_η is the restriction of an element $\alpha_0 \in F_{p+m}(X_\eta \setminus (Y_1^1 \cup Y_1^2))$.

Proof. The proof is by decreasing induction on $r, 1 \leq r \leq p-1$.

Suppose that $r = p-1$. Since $\{Y_r^{1,2}\}$ is a patching system with the freedom degree at least zero, we see that the natural maps $F_m(Y_\eta) \rightarrow F_m((Y_{p-1}^1)_\eta)$ and $F_m(Y_\eta) \rightarrow F_m((Y_{p-1}^2)_\eta)$ are equal to zero. Hence there are elements $\alpha_{p-1}^i \in F_{m+1}((Y_{p-1}^i)_\eta \setminus Y), i = 1, 2$ such that their coboundary is equal to $\alpha \in F_m(Y_\eta)$. The localization sequence associated to the closed embedding $(Y_{p-1}^1 \cap Y_{p-1}^2) \hookrightarrow Y_{p-1}^1 \cup Y_{p-1}^2$ implies that both elements α_{p-1}^1 and α_{p-1}^2 are restrictions of an element $\alpha_{p-1} \in F_{m+1}((Y_{p-1}^1 \cup Y_{p-1}^2)_\eta)$.

The induction step from $r+1$ to $r, 1 \leq r < p-1$ is analogous to the case $r = p-1$ with the subvarieties Y and $Y_{p-1}^{1,2}$ replaced by the subvarieties $Y_{r+1}^1 \cup Y_{r+1}^2$ and $Y_r^{1,2}$, respectively. At the end, for $r = 0$, we repeat the same with Y and $Y_{p-1}^{1,2}$ replaced by $Y_1^1 \cup Y_1^2$ and X , respectively. \square

Remark 3.45. The condition from Claim 3.44 is satisfied for all $\eta \in X^{(p)}$. Indeed, in this case one puts $\alpha = f_\eta \in F_m(k(\eta)) = F_m(Y_\eta)$.

Definition 3.46. Let $\{f_y\} \in \text{Gers}(X, F_*, p+m)^p$ be a cocycle in the Gersten complex, and $\{Y_r^{1,2}\}, 1 \leq r \leq p-1$ be a patching system on X for the support Y of $\{f_y\}$ with the freedom degree at least zero; then a cocycle $[\{f_y\}] \in \mathbf{A}(X, \mathcal{F}_{p+m}^X)^p$ is called a *good cocycle* for $\{f_y\} \in \text{Gers}(X, F_*, p+m)^p$ with respect to the patching system $\{Y_r^{1,2}\}, 1 \leq r \leq p-1$, if it satisfies all conditions from Proposition 3.42 and Claim 3.44 (for each point $\eta \in Y$).

It follows from Proposition 3.42 and Claim 3.44 that good cocycles always exist.

Claim 3.47. Let X be a smooth variety over an infinite perfect field; then for any cocycle $\{f_y\} \in \text{Gers}(X, F_*, m)^p$ in the Gersten complex and a patching system $\{Y_r^{1,2}\}, 1 \leq r \leq p-1$ on X for the support Y of $\{f_y\}$, there exists a good cocycle $[\{f_y\}] \in \mathbf{A}(X, \mathcal{F}_m^X)^p$ for $\{f_y\}$ with respect to the patching system $\{Y_r^{1,2}\}$.

The next technical lemma illustrates the freedom of choice in calculations with adeles.

Lemma 3.48. Let X be a smooth variety over an infinite perfect field and let the collection $\{f_y\} \in \text{Gers}(X, F_*, m)^p$ be supported on an equidimensional subvariety $Y \subset X$. Suppose that $d\{\tilde{f}_y\} = \{f_y\}$, where $\{\tilde{f}_y\} \in \text{Gers}(X, F_*, m)^{p-1}$. Suppose that $f \in \mathbf{A}(X, \mathcal{F}_m^X)^p$ is such that $\nu_X(f) = \{f_y\}$ and $f_U = 0$, where $f_U \in \mathbf{A}(U, \mathcal{F}_m^U)^p$ is the restriction of f to $U = X \setminus Y$. Let $\{Y_r^{1,2}\}$ be a patching system on X for Y ; then there exists an adèle $\tilde{f} \in \mathbf{A}(X, \mathcal{F}_m^X)^{p-1}$ such that $d\tilde{f} = f$, $\nu_X(\tilde{f}) = \{\tilde{f}_y\}$ and \tilde{f}_U is a good cocycle on U for $\{\tilde{f}_y\}_U \in \text{Gers}(U, F_*, m)^{p-1}$ with respect to the restriction of the patching system $\{Y_r^{1,2}\}$ to U .

Proof. We put $B^\bullet = \text{Ker}(\nu_X : \mathbf{A}(X, \mathcal{F}_m^X)^\bullet \rightarrow \text{Gers}(X, F_*, m)^\bullet)$. Since by Lemma 2.23, ν_X is surjective, we see that the complex B^\bullet is exact by Theorem 3.34. Let $\tilde{f}_1 \in \mathbf{A}(X, \mathcal{F}_m^X)^{p-1}$ be such that $\nu_X(\tilde{f}_1) = \{f_{\tilde{x}}\}$. We have $d(d(\tilde{f}_1) - f) = 0$ and $\nu_X(d(\tilde{f}_1) - f) = 0$, hence there exists $h \in B^{p-1}$ such that $dh = d(\tilde{f}_1) - f$. The adele $\tilde{f}_2 = \tilde{f}_1 - h$ satisfies $d\tilde{f}_2 = f$, $\nu_X(\tilde{f}_2) = \{f_{\tilde{y}}\}$.

If $p = 1$, then the adele $(\tilde{f}_2)_U$ is a good cocycle for $\{f_{\tilde{y}}\}_U \in \text{Gers}(U, m)^{p-1}$ on U . Suppose that $p \geq 2$. Let $\tilde{f}_3 \in \mathbf{A}(U, \mathcal{F}_m^U)^{p-1}$ be a good cocycle for $\{f_{\tilde{y}}\}_U \in \text{Gers}(U, F_*, m)^{p-1}$ with respect to the restriction of the patching system $\{Y_r^{1,2}\}$ to U . We have $d_U(\tilde{f}_3 - (\tilde{f}_2)_U) = -f_U = 0$, $\nu_U(\tilde{f}_3 - (\tilde{f}_2)_U) = 0$. Therefore there exists $h \in B_U^{p-2}$ such that $d_U(h) = \tilde{f}_3 - (\tilde{f}_2)_U$. Here we put $B_U^\bullet = \text{ker}(\nu_U)$ and d_U denotes the differential in the adelic complex on U . Let $h' \in \mathbf{A}(X, \mathcal{F}_m^X)^{p-2}$ be the extension by zero of h from U to X , i.e., we put $h'_{\eta_0 \dots \eta_{p-2}} = h_{\eta_0 \dots \eta_{p-2}}$ if $\eta_0 \dots \eta_{p-2}$ is a flag on U and, otherwise, we put $h'_{\eta_0 \dots \eta_{p-2}} = 0$ (see Corollary 2.7(i)). It follows easily that $h' \in B^{p-2}$ and the restriction of h' from X to U is equal to h . Thus the adele $\tilde{f} = \tilde{f}_2 + dh' \in \mathbf{A}(X, \mathcal{F}_m^X)^{p-1}$ satisfies all needed conditions. \square

4 Applications to K -cohomology

4.1 Generalities on K -cohomology and K -adeles

Recall several standard facts on sheaves of K -groups and K -cohomology.

Consider a weak homology theory for Noetherian schemes given by $F_* = K'_*$. The corresponding Zariski homology sheaves will be denoted by \mathcal{K}'_n , $n \in \mathbb{Z}$. We put $\text{Gers}(X, n)^\bullet = \text{Gers}(X, K'_*, n)^\bullet$, i.e., $\text{Gers}(X, n)^p = \bigoplus_{\eta \in X^{(p)}} K_{n-p}(k(\eta))$. For a scheme

X and an integer $n \geq 0$, let \mathcal{K}_n^X be the sheaf associated to the presheaf given by the formula $U \mapsto K_n(U)$ for any open subset $U \subset X$, where $K_n(U) = \pi_{n+1}(BQ\mathcal{P}(U))$ and $\mathcal{P}(U)$ is the exact category of coherent locally free sheaves on U . The Zariski cohomology groups $H^\bullet(X, \mathcal{K}_n^X)$ are called the K -cohomology of X . Evidently, there is a morphism of sheaves $\mathcal{K}_n^X \rightarrow (\mathcal{K}'_n)^X$ for any $n \geq 0$, which is an isomorphism if X is regular and separated. The sheaf $\mathcal{K}^X = \bigoplus_{n \geq 0} \mathcal{K}_n^X$ is the sheaf of supercommutative associative rings.

Any morphism of schemes $f : X \rightarrow Y$ defines a homomorphism of sheaves of algebras $f^* : \mathcal{K}^Y \rightarrow f_* \mathcal{K}^X$.

For any integers $m, n \geq 0$, there is a morphism of complexes of sheaves $\underline{\text{Gers}}(X, m)^\bullet \otimes \mathcal{K}_n^X \rightarrow \underline{\text{Gers}}(X, m+n)^\bullet$ given by the formula $\{f_\eta\} \otimes g \mapsto \{f_\eta \cdot i_\eta^* g\}$, where i_η^* is the natural morphism of sheaves $i_\eta^* : \mathcal{K}_n^X \rightarrow (i_{\bar{\eta}})_* K_n(k(\eta))$, $i_{\bar{\eta}} : \bar{\eta} \hookrightarrow X$. Thus the complex of sheaves $\underline{\text{Gers}}(X)^\bullet = \bigoplus_{m \geq 0} \underline{\text{Gers}}(X, m)^\bullet$ is a right module over the sheaf of associative rings \mathcal{K}^X and

the natural morphism $\bigoplus_{m \geq 0} (\mathcal{K}'_m)^X \rightarrow \underline{\text{Gers}}(X)^\bullet$ is a homomorphism of \mathcal{K}^X -modules. For any proper morphism $f : X \rightarrow Y$ of irreducible schemes, there is a canonical morphism of complexes of sheaves $Rf_* \underline{\text{Gers}}(X)^\bullet[d] = f_* \underline{\text{Gers}}(X)^\bullet[d] \xrightarrow{f_*} \underline{\text{Gers}}(Y)^\bullet$, where $d = \dim(f) = \dim(X) - \dim(Y)$. The projection formula tells that this morphism is a

homomorphism of \mathcal{K}^Y -modules via the homomorphism $\mathcal{K}^Y \rightarrow f_*\mathcal{K}^X$. Therefore general properties of resolutions of sheaves imply the following fact.

Lemma 4.1. *Let $f : X \rightarrow Y$ be a proper morphism of schemes. Let $a_1 \in H^{p_1}(X, \underline{Gers}(X)^\bullet) = H^{p_1}(Y, Rf_*\underline{Gers}(X)^\bullet)$, $b_i \in H^{p_i}(Y, \mathcal{K}^Y)$, $2 \leq i \leq k$ be classes in K -cohomology groups such that their k -th higher product $m_k(a_1, b_2, \dots, b_k)$ is well defined, where we consider $Rf_*\underline{Gers}(X)^\bullet$ as a module over \mathcal{K}^Y via the homomorphism $\mathcal{K}^Y \rightarrow f_*\mathcal{K}^X \rightarrow Rf_*\mathcal{K}^X$. Then the higher products $m_k(a_1, f^*(b_2), \dots, f^*(b_k))$ and $m_k(f_*(a_1), b_2, \dots, b_k)$ are also well defined and there is an equality*

$$f_*(m_k(a_1, f^*(b_2), \dots, f^*(b_k))) = (-1)^{d(p_2 + \dots + p_k)} m_k(f_*(a_1), b_2, \dots, b_k).$$

Remark 4.2. An alternative, more direct, way to show Lemma 4.1 for smooth varieties over an infinite perfect field is to use Theorem 3.34 and the adelic projection formula from Proposition 2.24.

Remark 4.3. Let us recall that Massey higher products for a right DG-module M^\bullet over a DG-ring A^\bullet are defined via the higher differentials in the spectral sequence associated with the Hochschild bicomplex $(M^\bullet \otimes (A^\bullet)^{\otimes(p-1)})^q$. More precisely, Massey higher products have the form

$$m_k : (H^{i_1}(M^\bullet) \otimes H^{i_2}(A^\bullet) \otimes \dots \otimes H^{i_k}(A^\bullet))^\circ \rightarrow {}^\circ(H^{i_1 + \dots + i_k - k}(M^\bullet)),$$

where for a group G , the notation $(G)^\circ$ means that we take a certain subgroup in G and ${}^\circ(G)$ means that we take a certain quotient of G . In particular, for a sheaf of associative algebras \mathcal{A} on a topological space X and a sheaf \mathcal{M} of right modules over \mathcal{A} , there are Massey higher products in cohomology groups $H^\bullet(X, \mathcal{A})$ and $H^\bullet(X, \mathcal{M})$; to define them one should take multiplicative resolutions for sheaves \mathcal{A} and \mathcal{M} on X (e.g., Godement resolutions), see more details in [6].

If X is a regular scheme of finite type over a field, then $\mathcal{K}_n^X = (\mathcal{K}'_n)^X$, the complex of sheaves $\underline{Gers}(X, n)^\bullet$ is quasiisomorphic to \mathcal{K}_n^X , and $H^n(X, \mathcal{K}_n^X) = CH^n(X)$ for any $n \geq 0$ (see [22] and also Proposition 3.8).

By Section 2.1, the complex $\mathbf{A}(X, \mathcal{K}^X)^\bullet$ is a DG-ring, any morphism of schemes $f : X \rightarrow Y$ defines a DG-homomorphism $\mathbf{A}(Y, \mathcal{K}^Y)^\bullet \rightarrow \mathbf{A}(X, \mathcal{K}^X)^\bullet$, and the complex $\underline{Gers}(X)^\bullet$ is a right DG-module over the DG-ring $\mathbf{A}(X, \mathcal{K}^X)^\bullet$. By Proposition 2.24, for any proper morphism $f : X \rightarrow Y$ of irreducible schemes, the morphism $\underline{Gers}(X)^\bullet[d] \rightarrow \underline{Gers}(Y)^\bullet$ is a homomorphism of right DG-modules over $\mathbf{A}(Y, \mathcal{K}^Y)^\bullet$ via the homomorphism $\mathbf{A}(Y, \mathcal{K}^Y)^\bullet \rightarrow \mathbf{A}(X, \mathcal{K}^X)^\bullet$, where $d = \dim(f)$.

Remark 4.4. It seems that there is no way to define a direct image map on the adelic complexes $\mathbf{A}(X, \mathcal{K}^X)^\bullet$ for proper morphisms of smooth varieties. This fact can be already seen in the simplest cases of finite morphisms or a closed embeddings. Nevertheless it is expected that there exists a *complete* version of K -adeles such that the complete adelic complex would have a (non-canonical) direct image map. Also, completed K -adeles should correspond to the global class field theory of arithmetical schemes, see [20]. Some particular cases were treated in [18]. However the “complete” theory is still to be built.

Remark 4.5. It follows from what is said above that for each $p \geq 0$, there is a canonical map α_p from $H^p(\mathbf{A}(X, \mathcal{K}_p^X)^\bullet)$ to the bivariant Chow group $A^p(X \xrightarrow{id} X)$ (see [8]). In addition, the natural map $\beta_p : H^p(X, \mathcal{K}_p) \rightarrow A^p(X \xrightarrow{id} X)$ factors through α_p .

Question 4.6. *Does there exist a singular variety X such that the image $\text{Im}(\alpha_p : H^p(\mathbf{A}(X, \mathcal{K}_p^X)^\bullet) \rightarrow A^p(X \xrightarrow{id} X))$ is strictly bigger than the image $\text{Im}(\beta_p : H^p(X, \mathcal{K}_p) \rightarrow A^p(X \xrightarrow{id} X))$, i.e., such that adelic cocycles define new elements in the bivariant Chow groups?*

Now let us fix an infinite perfect field k and consider K'_* as an l.a.f. homology theory over k , see Example 3.11, 1). Let X be an irreducible smooth variety over k ; then by Proposition 3.42 and Claim 3.44, for any algebraic cycle $Y = \sum n_i Y_i$ of codimension p on X , there is a good cocycle $[Y] = [\{1_Y\}] \in \mathbf{A}(X, \mathcal{K}_p^X)^p$, where $\{1_Y\}$ denotes a collection from $\bigoplus_{\eta \in X^{(p)}} \mathbb{Z}$ that equals $n_i \in \mathbb{Z}$ at the generic point η_i of Y_i for each i and equals $0 \in \mathbb{Z}$

at all other schematic points $\eta \in X^{(p)}$. Let us give two examples for adelic classes of subvarieties.

Let D be a (not necessary reduced or effective) divisor on X , $d = \dim(X)$. For each schematic point $\eta \in X$, consider a local equation $s_\eta \in k(X)^*$ of D in $X_\eta = \text{Spec}(\mathcal{O}_{X,\eta})$. Evidently, $s_\xi/s_\eta \in \mathcal{O}_{X,\eta}^*$ whenever $\xi \in \bar{\eta}$. Thus we get a 1-cocycle $[D] \in \mathbf{A}(X, \mathcal{K}_1^X)^1$ such that the $(X\eta)$ -component of $[D]$ is s_η^{-1} for $\eta \neq X$ and the $(\eta\xi)$ -component of $[D]$ is s_η/s_ξ for $\eta \neq X$, $\xi \in \bar{\eta}$, $\xi \neq \eta$. By construction, the class of $[D]$ in $H^1(\mathbf{A}(X, \mathcal{K}_1^X)^\bullet) = CH^1(X)$ coincides with the class of D in the first Chow group under the map ν_X . In [9], [11], and Corollary 4.23 it is proved that the intersection product in Chow groups coincides up to sign with the natural product in the corresponding K -cohomology groups. Thus we get the following adelic formula for the intersection index of divisors D_1, \dots, D_d when X is proper:

$$\begin{aligned} (D_1, \dots, D_d) &= - \sum_{\eta_0 \dots \eta_d} [k(\eta_d) : k] \nu_{\eta_0 \dots \eta_d} \{s_{1,\eta_1}^{-1}, s_{2,\eta_1}/s_{2,\eta_2}, \dots, s_{d,\eta_{d-1}}/s_{d,\eta_d}\} = \\ &= - \sum_{\eta_0 \dots \eta_d} [k(\eta_d) : k] \nu_{\eta_0 \dots \eta_d} \{s_{1,\eta_1}^{-1}, s_{2,\eta_2}^{-1}, \dots, s_{d,\eta_d}^{-1}\}, \end{aligned}$$

where the last identity follows from reciprocity law. This formula was proved by different methods first for $d = 2$ in [21] and for arbitrary d in [16]. We generalize the explicit computations from [21] and [16] in the proof of Theorem 4.22.

The next example is the intersection of a 1-cycle C and a divisor D in the three-dimensional irreducible smooth variety X over k . We describe explicitly a 2-cocycle $[C]$ in the adelic complex $\mathbf{A}(X, \mathcal{K}_2^X)^\bullet$ that represents C . Let us choose an effective reduced divisor E with the following properties: for each schematic point $\eta \in X$ of codimension at least two in X , there exists an element $t_\eta \in K_2(k(X))$ and a subdivisor $E_\eta \subset E$ such that $\text{sing}(t_\eta) \subset E$ and $d_\eta(\nu_{XE_\eta}(t_\eta)) = C_\eta$. Recall that ν_{XE_η} denotes the residue map from $K_2(k(X))$ to the direct sum of multiplicative groups of fields of rational functions on all irreducible components of E_η (see Section 3.4), C_η is the restriction of C to X_η ,

and d_η is the differential in the local Gersten resolution on X_η . The existence of such divisor E follows from Proposition 3.14 and Remark 3.31. Further, we define the adeles f_{012} and f_{013} such that $f_{XE_\eta\eta} = t_\eta^{-1} \in K_2(k(X))$, where t_η is as above. We put all the other components of f_{012} and f_{013} to be any elements from $(\mathcal{K}_2^X)_\eta = K_2(\mathcal{O}_{X,\eta})$. For each flag $\eta\xi$ of type (23), we have

$$d_\eta(\nu_{XE_\eta}(t_\eta)/\nu_{XE_\xi}(t_\xi)) = 0,$$

hence there exists an element $t_{\eta\xi} \in K_2(k(X))$ such that

$$d_\eta(t_{\eta\xi}) = \nu_{XE_\eta}(t_\eta)/\nu_{XE_\xi}(t_\xi).$$

This defines the adele f_{023} . Finally, we see that for each flag $\mu\eta\xi$ of type (123), the product $f_{\mu\eta\xi} = f_{X\eta\xi}f_{X\mu\xi}^{-1}f_{X\mu\eta}$ belongs to $(\mathcal{K}_2^X)_\mu$ and is also an adele. Thus we have defined the cocycle $[C] = f \in \mathbf{A}(X, \mathcal{K}_2^X)^2$ such that $[C]$ represents the class of C in $H^2(\mathbf{A}(X, \mathcal{K}_2^X)^\bullet) = CH^2(X)$ with respect to the map ν_X . From this we get the following intersection formula when X is proper:

$$(D, C) = \sum_{\mu\eta\xi} [k(\xi) : k] \nu_{X\mu\eta\xi} \{s_\mu^{-1}, f_{\mu\eta\xi}\} = \sum_{\mu\eta\xi} [k(\xi) : k] \nu_{X\mu\eta\xi} \{s_\mu^{-1}, t_{\eta\xi}\},$$

where, as above, s_μ is the local equation of the divisor D at the point μ and $\mu\eta\xi$ ranges over all flags of type (123) on X . As in the previous case, the last equality follows from reciprocity law.

Further, let us indicate a link between the adelic complex $\mathbf{A}(X, \mathcal{K}_n^X)^\bullet$ and coherent adeles.

Proposition 4.7. *Let X be a smooth variety over a field k ; then for any $n \geq 0$, there is a natural morphism of complexes*

$$\mathrm{dlog} : \mathbf{A}(X, \mathcal{K}_n^X)^\bullet \rightarrow a(X, \Omega_X^n)^\bullet,$$

where $a(X, \Omega_X^n)^\bullet$ is the complex of rational coherent adeles (see [19] and [14], Proposition 5.2.1) and the local component of this morphism for a flag $\eta_0 \dots \eta_p$ is equal to the natural map $K_n(\mathcal{O}_{\eta_0}) \rightarrow \Omega_{\mathcal{O}_{\eta_0}/k}^n$ (see [2]).

Proof. Let us prove by induction on p that for any natural number $p \geq 0$, any subset $M \subset S(X)_p$, and any open subset $U \subset X$, the map $\mathrm{dlog} : \mathbf{A}(M, \mathcal{K}_n^U) \rightarrow a(M, \Omega_U^n)$ is well defined. Since the sheaf Ω_X^n is locally free, it is 1-pure and we may suppose that $X \setminus U = D$ is a divisor.

Suppose that $p = 0$. We have $\mathbf{A}(M, \mathcal{K}_n^U) = \prod_{\eta \in M} (\mathcal{K}_n^U)_\eta$ and $\mathbf{A}(M, \Omega_U^n) = \varprojlim_{l \geq 0} \prod_{\eta \in M} (\Omega_X^n(lD))_\eta$ and, by Lemma 4.8, we get the needed result. For $p > 0$, we have

$$a(M, \Omega_U^n) = \prod_{\eta \in P(X)} a(\eta M, (\Omega_U^n)_\eta) = \prod_{\eta \in P(X)} \varprojlim_V a(\eta M, \Omega_{U \cap V}^n),$$

where for each schematic point $\eta \in P(X)$ the limit is taken over all open subsets $V \subset X$ containing η (for the second equality we use that the adelic functor commutes with direct limits of quasicoherent sheaves). Since the same equality holds for the adelic groups for the sheaf \mathcal{K}_n^X , the induction step is proved. \square

The author is grateful to C. Soulé for explaining the proof of the following lemma.

Lemma 4.8. *Rational differential forms from the image of the map $K_n(k(X)) \rightarrow \Omega_{k(X)/k}^n$ have pole of order at most one along each irreducible divisor $D \subset X$.*

Proof. Let us recall the construction of the map $K_n(R) \rightarrow \Omega_{R/\mathbb{Z}}^n$ and its properties. There are universal classes $c_n \in \varprojlim H^n(GL_m(R), \Omega_{R/\mathbb{Z}}^n)$, where the limit is taken over $m \geq 0$; they define the canonical maps $K_n(R) \rightarrow H_n(GL(R), \mathbb{Z}) \xrightarrow{c_n} \Omega_{R/\mathbb{Z}}^n$. The map c_n is trivial on $H_n(GL_{n-1}(R), \mathbb{Z})$. Moreover, the composition $R^* \times \dots \times R^* \rightarrow H_1(GL_1(R), \mathbb{Z}) \times \dots \times H_1(GL_1(R), \mathbb{Z}) \rightarrow H_n(GL_n(R), \mathbb{Z}) \rightarrow H_n(GL(R), \mathbb{Z}) \xrightarrow{c_n} \Omega_{R/\mathbb{Z}}^n$ is given by the formula $(r_1, \dots, r_n) \mapsto \frac{dr_1}{r_1} \wedge \dots \wedge \frac{dr_n}{r_n}$. Since one may suppose that $\dim X > 0$, the field $F = k(X)$ is infinite. By the results of Suslin, see [25], there is an isomorphism $H_n(GL_n(F), \mathbb{Z}) \cong H_n(GL(F), \mathbb{Z})$ and the natural map constructed above $F^* \times \dots \times F^* \rightarrow H_n(GL_n(F), \mathbb{Z})$ induces an isomorphism $K_n^M(F) \cong H_n(GL_n(F), \mathbb{Z})/H_n(GL_{n-1}(F), \mathbb{Z})$. Since for any non-zero rational functions $f_1, \dots, f_n \in k(X)^*$, the differential form $\frac{df_1}{f_1} \wedge \dots \wedge \frac{df_n}{f_n}$ has pole of order at most one along each irreducible divisor $D \subset X$, the lemma is proved. \square

Remark 4.9. It follows from [25] that for any field F the natural composition $K_n^M(F) \rightarrow K_n(F) \rightarrow \Omega_{F/\mathbb{Z}}^n$ is given by the formula $\{f_1, \dots, f_n\} \mapsto (-1)^n(n-1)! \frac{df_1}{f_1} \wedge \dots \wedge \frac{df_n}{f_n}$.

Remark 4.10. There is an equality $\mathrm{dlog}(f \cdot g) = -\frac{(m+n-1)!}{(m-1)!(n-1)!} \mathrm{dlog}(f) \cdot \mathrm{dlog}(g)$, where $f \in \mathbf{A}(X, \mathcal{K}_m^X)^\bullet$, $g \in \mathbf{A}(X, \mathcal{K}_n^X)^\bullet$, and in the right hand side we consider the product in the DG-ring $\bigoplus_{n \geq 0} a(X, \Omega_X^n)$.

Remark 4.11. Let Y be an algebraic cycle of codimension p on a smooth variety X over an infinite perfect field k . Then there is an explicit construction for the class of Y in the rational adelic group $a(X, \Omega_X^p)^p$. Indeed, one should take the image under the map dlog of the explicit (good) class $[Y]$ of Y in $\mathbf{A}(X, \mathcal{K}_p^X)^p$ constructed in Proposition 3.42.

4.2 Euler characteristic with support for K -groups

The construction and the results of this section are needed for the proof of Theorem 4.22 given in the next section. These results are not new; for example, they follow from Waldhausen K -theory of perfect complexes, developed in [26] or they may be obtained by using R -spaces constructed in [3]. However the author did not find a reference for an explicit construction, that is why this section is written.

By ΩS denote the loop space of a pointed space (S, s_0) . Let $f : (X, x_0) \rightarrow (Y, y_0)$ be a continuous map of pointed topological spaces. Consider the mapping path fibration

$$M(f) = \{(x, \varphi) | x \in X, \varphi : I \rightarrow Y, \varphi(0) = f(x)\},$$

where I is the interval $[0, 1]$. Recall that the homotopy fiber $F(f)$ is the fiber over y_0 of the natural map $M(f) \rightarrow Y$, $(x, \varphi) \mapsto \varphi(1)$. Notice that $F(f)$ and $M(f)$ are pointed spaces with the point (x_0, φ_0) , where φ_0 is the constant map to y_0 . There is a natural map $\Omega Y \rightarrow F(f)$, defined by $\gamma \mapsto (x_0, \gamma)$. Moreover, the composition $\Omega X \rightarrow \Omega Y \rightarrow F(f)$ is canonically homotopic to the constant map to $(x_0, \varphi_0) \in F(f)$. Indeed, the homotopy

$$G : \Omega X \times I \rightarrow F(f)$$

is given by

$$(\gamma, t) \mapsto (\gamma(t), \varphi_t),$$

where $\varphi_t(s) = (f \circ \gamma)(t + s(1 - t))$.

Let \mathcal{M} be an exact category, \mathcal{E}_3 be the exact category of exact triples of objects in \mathcal{M} . The exact functors

$$\{0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0\} \mapsto (M', M'')$$

and

$$(M', M'') \mapsto \{0 \rightarrow M' \rightarrow M' \oplus M'' \rightarrow M'' \rightarrow 0\}$$

induce the maps $BQ\mathcal{E}_3 \rightarrow BQ\mathcal{M} \times BQ\mathcal{M}$ and $BQ\mathcal{M} \times BQ\mathcal{M} \rightarrow BQ\mathcal{E}_3$, respectively. The well-known result of Quillen says that these two maps of pointed spaces are homotopy inverse (see [22], Theorem 2). Furthermore, let \mathcal{M}' and \mathcal{M}'' be two exact subcategories in \mathcal{M} and let \mathcal{E}'_3 be the category of exact triples in \mathcal{M} such that in the above notations the object M' is in \mathcal{M}' and the object M'' is in \mathcal{M}'' . Then, analogously, $BQ\mathcal{E}'_3$ is homotopy equivalent to $BQ\mathcal{M}' \times BQ\mathcal{M}''$.

In what follows we suppose for simplicity that \mathcal{M} is an abelian category (which is enough for further applications).

Lemma 4.12. *Let \mathcal{C}_n be the exact category of length n complexes of objects in \mathcal{M} and let \mathcal{E}_n be the full subcategory in \mathcal{C}_n consisting of all exact complexes. We put $B^i = \text{Im}(M^{i-1} \rightarrow M^i)$ for a complex M^\bullet . Then the natural maps $BQ\mathcal{C}_n \rightarrow BQ\mathcal{M}^{n+1}$, $BQ\mathcal{E}_n \rightarrow BQ\mathcal{M}^n$ induced by the exact functors*

$$\{0 \rightarrow M^0 \rightarrow \dots \rightarrow M^n \rightarrow 0\} \mapsto (M^0, \dots, M^n),$$

$$\{0 \rightarrow M^0 \rightarrow \dots \rightarrow M^n \rightarrow 0\} \mapsto (B^1, \dots, B^n),$$

respectively, are homotopy equivalences. Moreover, the following diagram of pointed spaces is commutative up to homotopy:

$$\begin{array}{ccc} BQ\mathcal{E}_n & \longrightarrow & BQ\mathcal{M}^n \\ \downarrow & & \downarrow i \\ BQ\mathcal{C}_n & \longrightarrow & BQ\mathcal{M}^{n+1}, \end{array}$$

where the horizontal maps are as defined above, the left vertical arrow is the natural inclusion, and i is induced by the exact functor

$$(B^1, \dots, B^n) \mapsto (B^1, B^1 \oplus B^2, \dots, B^{n-1} \oplus B^n, B^n).$$

Proof. We follow the proof of Theorem 1.11.7 from [26]. Nevertheless we do not use the language of K -theory spectra of Waldhausen categories.

The proof is by induction on $n \geq 3$. The case $n = 3$ is the result of Quillen mentioned above. For arbitrary $n \geq 4$, consider the natural inclusions of categories $\mathcal{E}_{n-1} \hookrightarrow \mathcal{E}_n$ and $\mathcal{M} \hookrightarrow \mathcal{E}_n$ given by the functors

$$\{0 \rightarrow M^0 \rightarrow \dots \rightarrow M^{n-1} \rightarrow 0\} \mapsto \{0 \rightarrow M^0 \rightarrow \dots \rightarrow M^{n-1} \rightarrow 0 \rightarrow 0\}$$

and

$$M \mapsto \{0 \rightarrow 0 \dots \rightarrow 0 \rightarrow M \rightarrow M \rightarrow 0\},$$

respectively. The category \mathcal{E}_n is equivalent to the category of exact triples in \mathcal{E}_n that start with an object from $\mathcal{E}_{n-1} \hookrightarrow \mathcal{E}_n$ and end with an object in $\mathcal{E}_2 = \mathcal{M} \hookrightarrow \mathcal{E}_n$. Indeed, the explicit equivalence is given by the functor

$$M^\bullet \mapsto \{0 \rightarrow \tau_{\leq(n-1)}(M^\bullet) \rightarrow M^\bullet \rightarrow \{0 \rightarrow B^n \rightarrow B^n \rightarrow 0\}\},$$

where $\tau_{\leq i}$ is the usual truncation functor associated to the canonical filtration on complexes. Thus, applying the result of Quillen modified above, we get that $BQ\mathcal{E}_n$ is homotopy equivalent to $BQ\mathcal{E}_{n-1} \times BQ\mathcal{E}$. Combining the explicit view of this homotopy and the inductive hypothesis, we get the desired result for \mathcal{E}_n .

The analogous reasoning leads to the needed result for \mathcal{C}_n . In this case we should replace the canonical filtration on complexes by the “bête” filtration and consider the inclusion of categories $\mathcal{M} \hookrightarrow \mathcal{C}_n$ given by the functor

$$M \mapsto \{0 \rightarrow \dots \rightarrow 0 \rightarrow M \rightarrow 0\}.$$

Finally, for any exact complex M^\bullet from \mathcal{E}_n , we have the exact sequences

$$0 \rightarrow B^i \rightarrow M^i \rightarrow B^{i+1} \rightarrow 0$$

for all $0 < i < n$. It follows from the proof of Corollary 1, §3, [22] that this leads to the needed homotopy equivalence in the diagram from the lemma. \square

Let F be the homotopy fiber of the natural map $BQ\mathcal{E}_n \rightarrow BQ\mathcal{C}_n$ and put $\mathbf{KN} = \Omega BQ\mathcal{N}$ for any exact category \mathcal{N} .

Corollary 4.13. *The inclusion of the categories $\mathcal{M} \hookrightarrow \mathcal{C}_n$ given by the functor $M \mapsto \{0 \rightarrow M \rightarrow 0 \rightarrow \dots \rightarrow 0\}$ induces the map $\mathbf{KM} \rightarrow \mathbf{KC}_n \rightarrow F$ such that the composition is a homotopy equivalence.*

Proof. Let us compute the induced map on homotopy groups. By Lemma 4.12, for each $i \geq 0$, there is a commutative diagram

$$\begin{array}{ccc} \pi_i(\mathbf{KE}_n) & \longrightarrow & \pi_{i+1}(BQM)^n \\ \downarrow & & \downarrow i_* \\ \pi_i(\mathbf{KC}_n) & \longrightarrow & \pi_{i+1}(BQM)^{n+1}, \end{array}$$

where the horizontal arrows are the isomorphisms induced by the maps described in Lemma 4.12. Thus there is a canonical isomorphism $\pi_i(F) \cong \pi_{i+1}(BQ\mathcal{M})$ given by the alternated sum of projections $\pi_{i+1}(BQ\mathcal{M})^{n+1} \rightarrow \pi_{i+1}(BQ\mathcal{M})$. Moreover, the composition $\pi_{i+1}(BQ\mathcal{M}) \cong \pi_i(\mathbf{K}\mathcal{M}) \rightarrow \pi_i(\mathbf{K}\mathcal{C}_n) \rightarrow \pi_i(F) \cong \pi_{i+1}(BQ\mathcal{M})$ is the identity map. Since by Milnor's result, $\mathbf{K}\mathcal{M}$ has the homotopy type of a CW-complex, we conclude by the well known theorem of Whitehead. \square

By $\mathcal{M}(S)$ denote the abelian category of coherent sheaves on a scheme S . We put $\mathcal{E}_n(S) = \mathcal{E}_n$, $\mathcal{C}_n(S) = \mathcal{C}_n$, $F(S) = F$, and $\mathbf{K}(S) = \mathbf{K}\mathcal{M}$ for $\mathcal{M} = \mathcal{M}(S)$.

Proposition 4.14. *Let S be a closed subscheme in the scheme T and let $\mathcal{C}_n(T, S)$ be a full subcategory in $\mathcal{C}_n(T)$ consisting of complexes whose cohomology sheaves have support on S , i.e., complexes whose restriction to $T \setminus S$ is in $\mathcal{E}_n(T \setminus S)$. Then there exists a well defined up to homotopy the “Euler characteristic with support” map $\chi : \mathbf{K}\mathcal{C}_n(T, S) \rightarrow \mathbf{K}(S)$ with the following properties:*

(i) *the induced homomorphism $\chi_* : K_0(\mathcal{C}_n(T, S)) \rightarrow K'_0(S)$ is equal to*

$$[\mathcal{F}^\bullet] \mapsto \sum_{i=0}^n (-1)^i [H^i(\mathcal{F}^\bullet)],$$

where \mathcal{F}^\bullet is in $\mathcal{C}_n(T, S)$ (here we imply the canonical isomorphism $K'_0(S) \cong K'_0(\tilde{S})$, induced by the closed embeddings $S_{\text{red}} \hookrightarrow S$ and $\tilde{S}_{\text{red}} \hookrightarrow \tilde{S}$, where \tilde{S} is any closed subscheme in T such that $S_{\text{red}} = \tilde{S}_{\text{red}}$);

(ii) *χ commutes with the direct image under closed embeddings; namely consider a closed subscheme $i : T' \hookrightarrow T$ and put $S' = S \times_T T'$. Then the following diagram of pointed spaces is commutative up to homotopy:*

$$\begin{array}{ccc} \mathbf{K}\mathcal{C}_n(T, S) & \xrightarrow{\chi} & \mathbf{K}(S) \\ \uparrow i_* & & \uparrow i_* \\ \mathbf{K}\mathcal{C}_n(T', S') & \xrightarrow{\chi} & \mathbf{K}(S'); \end{array}$$

(iii) *χ commutes with the restriction to open subsets; namely consider an open subset $j : U \hookrightarrow T$ and put $V = S \times_T U$. Then the following diagram of pointed spaces is commutative up to homotopy:*

$$\begin{array}{ccc} \mathbf{K}\mathcal{C}_n(T, S) & \xrightarrow{\chi} & \mathbf{K}(S) \\ \downarrow j^* & & \downarrow j^* \\ \mathbf{K}\mathcal{C}_n(U, V) & \xrightarrow{\chi} & \mathbf{K}(V). \end{array}$$

Proof. The natural map $\mathbf{K}\mathcal{C}_n(T, S) \rightarrow F(T \setminus S)$, defined by the diagram

$$\begin{array}{ccc} \mathbf{K}\mathcal{C}_n(T, S) & \longrightarrow & \mathbf{K}\mathcal{E}_n(T \setminus S) \\ \downarrow & & \downarrow \\ \mathbf{K}\mathcal{C}_n(T) & \longrightarrow & \mathbf{K}\mathcal{C}_n(T \setminus S) \\ \downarrow & & \downarrow \\ F(T) & \longrightarrow & F(T \setminus S), \end{array}$$

is canonically homotopy trivial. Hence there is a well defined map

$$\mathbf{KC}_n(T, S) \rightarrow F\{F(T) \rightarrow F(T \setminus S)\}.$$

On the other hand, by Corollary 4.13 and by Quillen's localization lemma (see [22]), the diagram

$$\begin{array}{ccc} \mathbf{K}(T) & \longrightarrow & \mathbf{K}(T \setminus S) \\ \downarrow & & \downarrow \\ F(T) & \longrightarrow & F(T \setminus S) \end{array}$$

induces a homotopy equivalence $\mathbf{K}(S) \rightarrow F\{F(T) \rightarrow F(T \setminus S)\}$. This defines the map $\chi : \mathbf{KC}_n(T, S) \rightarrow \mathbf{K}(S)$ uniquely up to homotopy.

Now we prove (i), i.e., we compute explicitly χ_* on π_0 -groups. Consider a point $[\mathcal{F}^\bullet]$ in $\mathbf{KC}_n(T, S)$ corresponding to a loop in $BQC_n(T, S)$ defined by a complex \mathcal{F}^\bullet from $\mathcal{C}_n(T, S)$ in the standard way. There exists a homotopy inside $BQC_n(T, S)$ between the loop $[\mathcal{F}^\bullet]$ and the sum of loops

$$[\tau_{\leq(n-1)}\mathcal{F}^\bullet] + [\{0 \rightarrow \dots \rightarrow B^n \rightarrow B^n \rightarrow 0\}] + [H^n(\mathcal{F}^\bullet)[-n]].$$

In addition, $[H^n(\mathcal{F}^\bullet)[-n]]$ is homotopic inside $BQC_n(T, S)$ to the sum

$$(-1)^n[H^n(\mathcal{F}^\bullet)] + \sum_{j=0}^{n-1} (-1)^j[\{0 \rightarrow H^n(\mathcal{F}^\bullet) \rightarrow H^n(\mathcal{F}^\bullet) \rightarrow 0\}[-j]],$$

where the short complexes have support in degrees 0 and 1. Continuing, we show by induction that the initial loop may be homotoped inside $BQC_n(T, S)$ to the sum of $\sum (-1)^i[H^i(\mathcal{F}^\bullet)]$ and some loops in $BQE_n(T)$. The classes of points in $\mathbf{KC}_n(T, S)$, corresponding to loops in $BQE_n(T)$, evidently have zero image under χ_* on π_0 -groups, and we are done.

For the proof of (ii) and (iii) one uses that the natural maps

$$T' \setminus S' = (T \setminus S) \times_T T' \hookrightarrow T \setminus S$$

and

$$U \setminus V = (T \setminus S) \times_T U \hookrightarrow T \setminus S$$

are closed embedding and open embedding, respectively. Also, one uses the fact that if in the commutative diagram of pointed spaces

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

the vertical arrows are homotopy equivalences, then the diagram remains commutative up to homotopy after we take the homotopy inverse to the vertical homotopy equivalences. \square

Remark 4.15. A similar way to define the Euler characteristic with support on K -groups $K_*(\mathcal{C}_n(T, S)) \rightarrow K_*(S)$ is to use the R -space construction from [3], Section 1. Recall that for any exact category \mathcal{M} , the space $R\mathcal{M}$ has the same homotopy type as $BQ\mathcal{M}$ and the H -space $R\mathcal{M}$ has a canonical inverse map, not just an inverse up to homotopy. Thus there is the Euler characteristic map $RC_n(T, S) \rightarrow R\mathcal{M}(T)$ such that its composition with the natural map $R\mathcal{M}(T) \rightarrow R\mathcal{M}(T \setminus S)$ is canonically homotopy trivial. This gives a well defined map $RC_n(T, S) \rightarrow R\mathcal{M}(S)$ up to homotopy.

Now consider a complex \mathcal{P}^\bullet from $\mathcal{C}_n(T, S)$ such that \mathcal{P}^\bullet consists of flat sheaves on T . Let the map $* \cdot \mathcal{P}^\bullet : \mathbf{K}(T) \rightarrow \mathbf{K}(S)$ be the composition of the map induced by the exact functor $* \otimes_{\mathcal{O}_T} \mathcal{P}^\bullet : \mathcal{M}(T) \rightarrow \mathcal{C}_n(T, S), \mathcal{F} \mapsto \mathcal{F} \otimes_{\mathcal{O}_T} \mathcal{P}^\bullet$ and the map $\chi : \mathbf{K}\mathcal{C}_n(T, S) \rightarrow \mathbf{K}(S)$ from Proposition 4.14. The map $* \cdot \mathcal{P}^\bullet$ is well defined up to homotopy. A standard argument shows that the maps $* \cdot \mathcal{P}^\bullet$ and $* \cdot (\mathcal{P}')^\bullet$ are homotopy equivalent for quasiisomorphic complexes \mathcal{P}^\bullet and $(\mathcal{P}')^\bullet$. By Proposition 4.14(i), for any class $[\mathcal{F}]$ from $K'_0(T)$ we have $[\mathcal{F}] \cdot \mathcal{P}^\bullet = \sum (-1)^i H^i(\mathcal{F} \otimes_{\mathcal{O}_T} \mathcal{P}^\bullet) \in K'_0(S)$.

Proposition 4.16. *Let \mathcal{P}^\bullet be a finite flat resolution of \mathcal{O}_S on T (here the complex \mathcal{P}^\bullet has support in non-positive terms) and suppose that T admits an ample line bundle (for example, T is quasi-projective). Then the map $* \cdot \mathcal{P}^\bullet$ is homotopic to the map f^* , where $f : S \hookrightarrow T$ is the closed embedding (see [22], §7, 2.5).*

Proof. By Quillen resolution theorem (see [22], §4, Corollary 1), the space $BQ\mathcal{M}(T)$ is homotopy equivalent to its subspace $BQ\mathcal{F}(T)$, where $\mathcal{F}(T)$ is the exact category of coherent sheaves on T that are Tor-independent with \mathcal{O}_S . For $BQ\mathcal{F}(T)$ we have the exact sequence of functors from $\mathcal{F}(T)$ to $\mathcal{C}_n(T, S)$

$$0 \rightarrow \tau'_{\leq 0}(* \otimes_{\mathcal{O}_T} \mathcal{P}^\bullet) \rightarrow * \otimes_{\mathcal{O}_T} \mathcal{P}^\bullet \rightarrow * \otimes_{\mathcal{O}_T} \mathcal{O}_S \rightarrow 0$$

where we put

$$\tau'_{\leq 0}(A^\bullet) = \{\dots \rightarrow A^i \rightarrow \dots \rightarrow A^{-1} \rightarrow B^0 \rightarrow 0\},$$

for any complex A^\bullet . It follows from the proof of Corollary 1, §3, [22] that the map $\mathbf{K}(* \otimes_{\mathcal{O}_T} \mathcal{P}^\bullet)$ is homotopic to the map $\mathbf{K}(\tau'_{\leq 0}(* \otimes_{\mathcal{O}_T} \mathcal{P}^\bullet)) + \mathbf{K}(* \otimes_{\mathcal{O}_T} \mathcal{O}_S)$, where the sum is taken with respect to the natural H -structures on \mathbf{K} -spaces of exact categories. Moreover, the image of the map $\mathbf{K}(\tau'_{\leq 0}(* \otimes_{\mathcal{O}_T} \mathcal{P}^\bullet))$ is in the space $\mathbf{K}(\mathcal{E}_n(T)) \subset \mathbf{K}(\mathcal{C}_n(T, S))$ and the map $\mathbf{K}(* \otimes_{\mathcal{O}_T} \mathcal{O}_S)$ equals f^* on $\mathbf{K}(\mathcal{F}(T))$. This concludes the proof. \square

Remark 4.17. For the elements from $K_*(T)$, the map $* \cdot \mathcal{P}^\bullet$ is equal to the composition of the usual restriction to S with multiplication by $\chi_*([\mathcal{P}^\bullet]) \in K'_0(S)$.

Proposition 4.18. *Let $i : T' \hookrightarrow T$ be a closed embedding and put $S' = S \times_T T'$, $j : U = T \setminus T' \hookrightarrow T$, $V = S \times_T U$. Consider arbitrary elements $x \in K_m(S)$, $y \in K'_n(U)$, where $m, n \geq 0$, $m + n \geq 1$. Then we have*

$$\nu(x \cdot (y \cdot \mathcal{P}^\bullet)) = (-1)^m x \cdot (\nu(y) \cdot i^* \mathcal{P}^\bullet) \in K'_{m+n-1}(S'),$$

where $\nu : K'_*(U) \rightarrow K'_{*-1}(T')$ denotes the usual boundary map (the same for V and S').

Proof. By Proposition 4.14(ii),(iii), both squares in the following diagram of spaces are commutative up to homotopy:

$$\begin{array}{ccc}
\mathbf{K}(\mathcal{P}(S)) \wedge \mathbf{K}(T') & \xrightarrow{* \cdot \mathcal{P} \cdot \otimes *} & \mathbf{K}(S') \\
\downarrow i_* \times \text{id} & & \downarrow i_* \\
\mathbf{K}(\mathcal{P}(S)) \wedge \mathbf{K}(T) & \xrightarrow{* \cdot \mathcal{P} \cdot \otimes *} & \mathbf{K}(S) \\
\downarrow j^* \times \text{id} & & \downarrow j^* \\
\mathbf{K}(\mathcal{P}(S)) \wedge \mathbf{K}(U) & \xrightarrow{* \cdot \mathcal{P} \cdot \otimes *} & \mathbf{K}(V),
\end{array}$$

where $\mathcal{P}(S)$ denotes the exact category of locally free coherent sheaves on S and “ \wedge ” denotes the wedge product of pointed topological spaces. Thus rest is a direct application of Theorem 2.5 from [11]. \square

Let us mention the following simple fact.

Lemma 4.19. *Let $S \hookrightarrow T$ be a closed embedding, S_{red} be the reduced scheme, $j : S_{red} \rightarrow S$ be a natural embedding. Then $j_* \circ \nu_{red} = \nu$, where $\nu : K'_*(T \setminus S) \rightarrow K'_{*-1}(S)$, $\nu_{red} : K'_*(T \setminus S) \rightarrow K'_{*-1}(S_{red})$ are the boundary maps.*

Proof. This follows immediately from the commutativity of the diagram of CW-spaces

$$\begin{array}{ccc}
BQM(S_{red}) & \xrightarrow{j_*} & BQM(S) \\
\downarrow & & \downarrow \\
BQM(T) & \xrightarrow{=} & BQM(T) \\
\downarrow & & \downarrow \\
BQM(T \setminus S_{red}) & \xrightarrow{=} & BQM(T \setminus S)
\end{array}$$

after we pass to the long homotopy sequences, associated to the vertical sequences, which are fibrations up to homotopy. \square

Lemma 4.19 implies the following statement.

Corollary 4.20. *Proposition 4.18 remains true after we change the schematic intersection $S' = S \times_T T'$ by its reduced part S'_{red} .*

Example 4.21. Let k be a field, T be the local scheme $(\mathbb{A}_k^2)_{(0,0)}$ with coordinates (x, y) , $T' = \{xy = 0\}$, $S = \{x + y = 0\}$, $\mathcal{P}^\bullet = \{\mathcal{O}_T \xrightarrow{x+y} \mathcal{O}_T\}$. Let $f(x), g(y)$ be rational functions on the corresponding irreducible components of T' such that $f(x)$ and $g(y)$ have the opposite valuations at the origin. These functions naturally define an element $\alpha \in K'_1(T')$. We have $\alpha \cdot i^* \mathcal{P}^\bullet = a/b \in K_1(k)$, where a and b are the main parts of $f(x)$ and $g(y)$ in x and y , respectively.

4.3 Formula for product in K -cohomology

Let X be an irreducible smooth quasiprojective variety over an infinite perfect field, Y, Z be two equidimensional subvarieties in X of codimensions p and q , respectively. Consider two cocycles $\{f_y\} \in \bigoplus_{y \in Y^{(0)}} K_m(k(y))$ and $\{g_z\} \in \bigoplus_{z \in Z^{(0)}} K_n(k(z))$ in the Gersten complexes $Gers(X, p+m)^p$ and $Gers(X, q+n)^q$, respectively.

Suppose that the subvarieties Y and Z intersect properly. In addition, suppose that for any irreducible component w of the intersection $W = Y \cap Z$, the collections $\{f_y\}_w$ and $\{g_z\}_w$ are represented by some elements α_w and β_w from the groups $K_m(Y_w)$ and $K'_n(Z_w)$, respectively (recall that the index w means the restriction of a collection to $X_w = \text{Spec}(\mathcal{O}_{X,w})$).

By Remark 3.31 and Remark 3.32, there are patching systems $\{Y_r^{1,2}\}$, $1 \leq r \leq p-1$ and $\{Z_s^{1,2}\}$, $1 \leq s \leq q-1$ on X for subvarieties Y and Z , respectively, with the freedom degree at least zero such that the following conditions are satisfied:

- (i) for any s , $1 \leq s \leq q-1$, no irreducible component in $Y \cap Z_s^1$ is contained in Z_s^2 ;
- (ii) each irreducible component in $Y_{p-1}^{1,2}$ contains some irreducible component in Y , each irreducible component in $Y_r^{1,2}$, $2 \leq r \leq p-2$, contains some irreducible component in $Y_{r+1}^1 \cup Y_{r+1}^2$, and the analogous is true for the patching system $Z_s^{1,2}$, $1 \leq s \leq q-1$; in particular, the subvarieties $Y_r^{1,2}$ and Y meet the subvarieties $Z_s^{1,2}$ and Z properly.

Let $f \in \mathbf{A}(X, \mathcal{K}_{p+m}^X)^p$ and $g \in \mathbf{A}(X, \mathcal{K}_{q+n}^X)^q$ be good cocycles for the collections $\{f_y\}$ and $\{g_z\}$ with respect to the patching systems $\{Y_r^{1,2}\}$, $1 \leq r \leq p-1$ and $\{Z_s^{1,2}\}$, $1 \leq s \leq q-1$, respectively, such that $\{Y_r^{1,2}\}$ and $\{Z_s^{1,2}\}$ satisfy two conditions from above (see Section 3.6).

Theorem 4.22. *Let $\mathcal{P}^\bullet \rightarrow \mathcal{O}_Y$ be a finite flat resolution of \mathcal{O}_Y on X . Under the above assumptions we have:*

$$\nu_X(f \cdot g) = (-1)^{(p+m)q} \{\bar{\alpha}_w \cdot (\beta_w \cdot i_Z^* \mathcal{P}^\bullet)\} \in \bigoplus_{w \in W^{(0)}} K_{m+n}(k(w)),$$

where $i_Z : Z \hookrightarrow X$ is the closed embedding (considered locally around w for each summand) and the bar over α_w denotes the image under the natural homomorphisms $K_m(Y_w) \rightarrow K_m(k(w))$ for each point w .

Proof. Let $\eta_0 \dots \eta_{p+q}$ be a flag of type $(0 \dots p+q)$ on X . By condition (i) from Proposition 3.42, we have $f_{\eta_0 \dots \eta_p} \cdot g_{\eta_p \dots \eta_{p+q}} = 0$ unless η_r is the generic point of an irreducible component of Y_r^1 for all r , $1 \leq r \leq p-1$, η_p is the generic point of an irreducible component of Y , η_{p+s} is the generic point of an irreducible component of the intersection $Y \cap Z_s^1$ for all s , $1 \leq s \leq q-1$, and η_{p+q} is the generic point of an irreducible component of the intersection $Y \cap Z$.

Suppose that the flag $\eta_0 \dots \eta_{p+q}$ enjoys this property. Combining the assumption on the patching systems $\{Y_r^{1,2}\}$, $1 \leq r \leq p-1$ and $\{Z_s^{1,2}\}$, $1 \leq s \leq q-1$, condition (ii) from Proposition 3.42, and Claim 3.44, we see that

$$f_{\eta_0 \dots \eta_p} = \tilde{f}_{\eta_p} \in K_{p+m}(X_{\eta_p} \setminus (Y_1^1 \cup Y_1^2)), g_{\eta_p \dots \eta_{p+q}} = \tilde{g}_{\eta_{p+q}} \in K_{q+n}(X_{\eta_{p+q}} \setminus (Z_1^1 \cup Z_1^2)),$$

where

$$d_{\eta_p} \nu_{XY_1^1 \dots Y_{p-1}^1}(\tilde{f}_{\eta_p}) = (-1)^{\frac{p(p+1)}{2}} f_{\eta_p} = \{f_y\}_{\eta_p},$$

$$d_{\eta_{p+q}} \nu_{XZ_1^1 \dots Z_{q-1}^1}(\tilde{g}_{\eta_{p+q}}) = (-1)^{\frac{q(q+1)}{2}} \{g_z\}_{\eta_{p+q}}.$$

Henceforth, $\nu_{0 \dots p}(f \cdot g)_{\eta_p \dots \eta_{p+q}} = \nu_{XY_1^1 \dots Y_{p-1}^1 \eta_p}(\tilde{f}_{\eta_p} \cdot \tilde{g}_{\eta_{p+q}})$.

We claim that the residue $\nu_{XY_1^1 \dots Y_{p-1}^1 \eta_p}(\tilde{f}_{\eta_p} \cdot \tilde{g}_{\eta_{p+q}})$ is equal to the product $(-1)^{\frac{p(p+1)}{2}} f_{\eta_p} \cdot i_{\eta_p}^* \tilde{g}_{\eta_{p+q}} \in K_{m+q+n}(k(\eta_p))$, where $i_{\eta_p} : \text{Spec}(k(\eta_p)) \rightarrow X$ is the natural morphism (notice that η_p does not belong to $Z_1^1 \cup Z_1^2$). This can be shown using Proposition 4.18 first with $S = T = X_{\eta_p}$, $\mathcal{P}^\bullet = \mathcal{O}_S$, $T' = (Y_1^1 \cup Y_1^2)_{\eta_p}$ and then, inductively, with $S = T = (Y_s^1)_{\eta_p}$, $\mathcal{P}^\bullet = \mathcal{O}_S$, $T' = (Y_{s+1}^1 \cup Y_{s+1}^2)_{\eta_p}$ for $1 \leq s \leq p-1$ (more precisely, for the induction step we use that the cocycles f and g satisfy the conditions from Claim 3.44).

In addition, the product $f_{\eta_p} \cdot i_{\eta_p}^* \tilde{g}_{\eta_{p+q}} \in K_{m+q+n}(k(\eta_p))$ equals the restriction to $\text{Spec}(k(\eta_p))$ of the product

$$\alpha_{\eta_{p+q}} \cdot i_Y^* \tilde{g}_{\eta_{p+q}} \in K_{m+q+n}(Y_{\eta_{p+q}} \setminus (Z_1^1 \cup Z_1^2)),$$

where $i_Y : Y \hookrightarrow X$ is the closed embedding. Consequently, we have $\nu_{p+q}(f \cdot g)_w = (-1)^{\frac{p(p+1)}{2}} \nu_{Y, Y \cap Z_1^1, \dots, Y \cap Z_{q-1}^1, w}(\alpha_w \cdot i_Y^* \tilde{g}_w)$ if w is the generic point of an irreducible component of the intersection $W = Y \cap Z$. Otherwise, $\nu_{p+q}(f \cdot g)_w = 0$.

Let w be the generic point of an irreducible component of W . In what follows we consider all schemes on X locally around the schematic point w , i.e., we consider their restrictions to $X_w = \text{Spec}(\mathcal{O}_{X,w})$, though we denote them by the same letter. Put $Z_0^1 = X$, $Z_q^1 = Z$ and by $i_s : Z_s^1 \hookrightarrow X$ denote the natural closed embedding for each s , $0 \leq s \leq q$. By Proposition 4.14 and Corollary 4.20, the following diagram commutes up to sign $(-1)^m$ for all s , $0 \leq s \leq q-1$:

$$\begin{array}{ccc} K'_{*+m}(Y \cap (Z_s^1 \setminus (Z_{s+1}^1 \cup Z_{s+1}^2))) & \longleftarrow & K'_*(Z_s^1 \setminus (Z_{s+1}^1 \cup Z_{s+1}^2)) \\ \downarrow & & \downarrow \\ K'_{*+m-1}(Y \cap (Z_{s+1}^1 \setminus Z_{s+1}^2)) & \longleftarrow & K'_{*-1}(Z_{s+1}^1 \setminus Z_{s+1}^2), \end{array}$$

where the vertical arrows are the compositions of the boundary maps with the restrictions to open subsets and the horizontal arrows are the compositions of the map $* \cdot i_s^* \mathcal{P}^\bullet$ (respectively, $* \cdot i_{s+1}^* \mathcal{P}^\bullet$) with the multiplication on the right by the restriction of $\alpha_{\eta_{p+q}} \in K_m(Y)$ to the corresponding closed subsets in Y . Therefore, by Claim 3.44 and Remark 4.17, we get

$$\begin{aligned} \nu_{Y, Y \cap Z_1^1, \dots, Y \cap Z_{q-1}^1, w}(\alpha_w \cdot i_Y^* \tilde{g}_w) &= \nu_{Y, Y \cap Z_1^1, \dots, Y \cap Z_{q-1}^1, w}(\alpha_w \cdot (\tilde{g}_w \cdot \mathcal{P}^\bullet)) = \\ &= \bar{\alpha}_w \cdot ((-1)^{mq} \nu_{XZ_1^1 \dots Z_{q-1}^1 w}(\tilde{g}_w) \cdot i_Z^* \mathcal{P}^\bullet) = (-1)^{mq + \frac{q(q+1)}{2}} \bar{\alpha}_w \cdot (\beta_w \cdot i_Z^* \mathcal{P}^\bullet). \end{aligned}$$

Combining this with the equality $\nu_X = (-1)^{\frac{(p+q)(p+q+1)}{2}} \nu_{p+q}$, we conclude the proof of the theorem. \square

Let $\{f_y\} \in Gers(X, p+m)^p$ and $\{g_z\} \in Gers(X, q+n)^q$ be two Gersten cocycles as above with one additional property: for any irreducible component w of the intersection $W = Y \cap Z$, the collection $\{g_z\}_w$ is represented by an element β_w from the group $K_n(Z_w)$.

Corollary 4.23. *Under the above assumptions, the product of the classes of $\{g_y\}$ and $\{h_z\}$ in K -cohomology groups is represented by the Gersten cocycle*

$$(-1)^{(p+m)q} \{(Y, Z; w) \overline{\alpha}_w \cdot \overline{\beta}_w\} \in \bigoplus_{w \in W^{(0)}} K_{m+n}(k(w)),$$

where $(Y, Z; w)$ is the local intersection index of Y and Z at the component w . In particular, the intersection product in Chow groups coincides up to sign with the natural product in the corresponding K -cohomology groups (the last assertion had been proved by different methods in [9] and [11]).

Proof. The composition of the morphisms of complexes $\mathcal{K}_n(\mathcal{O}_X) \rightarrow \underline{\mathbf{A}}(X, \mathcal{K}_n)^\bullet \xrightarrow{\nu_X} \underline{Gers}(X, n)^\bullet$ is identity on the (hyper)cohomology groups. Therefore the product of the classes of Gersten cocycles in K -cohomology groups is represented by the image under the map ν_X of the adelic product of the corresponding adelic cocycles.

On the other hand, we have $(Y, Z; w) = \sum_{i \geq 0} (-1)^i l(\text{Tor}_i^{\mathcal{O}_{X,w}}(\mathcal{O}_{Y,w}, \mathcal{O}_{Z,w}))$, where $l(\cdot)$

is the length of an $\mathcal{O}_{X,w}$ -module, i.e., the length of a filtration whose adjoint quotients are one-dimensional vector spaces over the field $k(w)$. Thus the lemma follows directly from Theorem 4.22 and Remark 4.17. \square

Remark 4.24. If each generic point w of an irreducible component of the intersection $W = Y \cap Z$ is regular on Y and Z , then the conditions of Corollary 4.23 are satisfied.

4.4 Massey triple product and the Weil pairing

In this section we apply the adelic resolution to computation of some Massey triple product. Let X be a smooth variety of dimension d over an infinite perfect field k . Consider elements $\alpha \in CH^p(X)_l = H^p(X, \mathcal{K}_p^X)_l$, $\beta \in CH^q(X)_l = H^q(X, \mathcal{K}_q^X)_l$, and $l \in H^0(X, \mathcal{K}_0^X) = \mathbb{Z}$ such that $p + q = d + 1$. The triple (α, l, β) satisfies $\alpha \cdot l = l \cdot \beta = 0$ in K -cohomology groups, hence there is a triple product

$$m_3(\alpha, l, \beta) \in H^d(X, \mathcal{K}_{d+1}^X) / (\alpha \cdot H^{q-1}(X, \mathcal{K}_q^X) + H^{p-1}(X, \mathcal{K}_p^X) \cdot \beta).$$

We compute this product explicitly. Let us represent the classes $\alpha \in CH^p(X)_l$ and $\beta \in CH^q(X)_l$ by cycles $Y = \sum_i m_i \cdot Y_i$ and $Z = \sum_j n_j \cdot Z_j$, respectively, where Y_i and Z_j are irreducible subvarieties in X of codimensions p and q , respectively, and $m_i, n_j \in \mathbb{Z}$. Since $p + q = d + 1$, it can be assumed that the supports $|Y| = \cup_i Y_i$ and $|Z| = \cup_j Z_j$ do not intersect. Let $\{f_{\tilde{y}}\} \in Gers(X, p)^{p-1}$ and $\{g_{\tilde{z}}\} \in Gers(X, q)^{q-1}$ be two collections such that $d\{f_{\tilde{y}}\} = lY$, $d\{g_{\tilde{z}}\} = lZ$ and let $\tilde{Y} \subset X$ and $\tilde{Z} \subset X$ be the supports of $\{f_{\tilde{y}}\}$ and $\{g_{\tilde{z}}\}$, respectively. It follows the moving lemma for higher Chow groups (see [4] and also [15]) that for X either affine or projective, one can choose $\{f_{\tilde{y}}\}$ and $\{g_{\tilde{z}}\}$ such that

the intersections $\tilde{Y} \cap |Z|$ and $|Y| \cap \tilde{Z}$ are proper and the rational functions $f_{\tilde{y}}$ and $g_{\tilde{z}}$ are regular at all points from the intersections $\tilde{Y} \cap |Z|$ and $|Y| \cap \tilde{Z}$, respectively. For each point $x \in \tilde{Y} \cap |Z|$, we put $f(x) = \prod_{\tilde{y}} f_{\tilde{y}}^{(\tilde{y}, Z; x)}(x)$. Similarly, we define $g(x)$ for each point $x \in |Y| \cap \tilde{Z}$.

Proposition 4.25. *Under the above assumptions, the triple product $m_3(\alpha, l, \beta)$ is represented by the Gersten cocycle*

$$(-1)^{pq} \left(\sum_{x \in \tilde{Y} \cap |Z|} f(x) \cdot x + \sum_{x \in |Y| \cap \tilde{Z}} g^{-1}(x) \cdot x \right) \in \text{Gers}(X, d+1)^d.$$

Proof. By Remark 3.31 and Remark 3.32, there are patching systems $\{Y_r^{1,2}\}$, $\{\tilde{Y}_r^{1,2}\}$, $\{Z_s^{1,2}\}$, and $\{\tilde{Z}_s^{1,2}\}$ on X for subvarieties $|Y|$, \tilde{Y} , $|Z|$, and \tilde{Z} , respectively, such that the pairs of patching systems $(\{Y_r^{1,2}\}, \{\tilde{Y}_r^{1,2}\})$ and $(\{\tilde{Y}_r^{1,2}\}, \{Z_s^{1,2}\})$ satisfy conditions of Theorem 4.22. Let $[Y] \in \mathbf{A}(X, \mathcal{K}_p)^p$ and $[Z] \in \mathbf{A}(X, q)^q$ be good cocycles for Y and Z with respect to the patching systems $\{Y_r^{1,2}\}$ and $\{Z_s^{1,2}\}$ on X , respectively. By Lemma 3.48, there are adeles $f \in \mathbf{A}(X, \mathcal{K}_p^X)^{p-1}$ and $g \in \mathbf{A}(X, \mathcal{K}_q^X)^{q-1}$ such that $df = l[Y]$, $dg = l[Z]$, and the restrictions f_U and g_V are good cocycles for $\{f_{\tilde{y}}\}_U$ and $\{g_{\tilde{z}}\}_V$ with respect to the patching systems $\{\tilde{Y}_r^{1,2}\}_U$ and $\{\tilde{Z}_s^{1,2}\}_V$ respectively, where $U = X \setminus |Y|$ and $V = X \setminus |Z|$. By definition, $m_3(\alpha, l, \beta)$ is represented by the Gersten cocycle $\nu_X(f \cdot [Z] - (-1)^p [Y] \cdot g)$. Since $\nu_X(f \cdot [Z]) = \nu_U((f \cdot [Z])_U)$ and $\nu_X([Y] \cdot g) = \nu_V([Y] \cdot g)_V$, we conclude by Theorem 4.22 and Corollary 4.23. \square

Let X be a smooth projective variety of dimension d over a field k . Evidently, for any degree zero element $\alpha \in CH^d(X)$, we have $\pi_*(\alpha \cdot H^0(X, \mathcal{K}_1^X)) = \pi_*(\alpha \cdot \mathcal{O}^*(X)) = 1$, where $\pi_* : H^d(X, \mathcal{K}_{d+1}^X) \rightarrow k^*$ is the direct image map associated with the structure morphism $\pi : X \rightarrow \text{Spec}(k)$.

Proposition 4.26. *The subgroup $H^{d-1}(X, \mathcal{K}_d^X) \cdot \text{Pic}^0(X) \subset H^d(X, \mathcal{K}_{d+1}^X)$ is in the kernel of the direct image map $\pi_* : H^d(X, \mathcal{K}_{d+1}^X) \rightarrow k^*$.*

Proof. After the base change, one may assume that the ground field k is algebraically closed. Consider a K_1 -chain $\{f_j\} \in \text{Gers}(X, d)^{d-1}$ with $\sum_j \text{div}(f_j) = 0$ and a group homomorphism $\text{Pic}^0(X)(k) \rightarrow k^*$ given by $\beta \mapsto \pi_*(\{f_j\} \cdot \beta)$. We claim that this homomorphism is induced by a regular morphism from the Picard variety $\text{Pic}^0(X)$ to the algebraic group \mathbb{G}_m and therefore is identically equal to $1 \in k^*$.

For each j , by C'_j denote the complement to the divisor of the function f_j on C_j . For any closed point $\gamma \in \text{Pic}^0(X)$, there exists a rational section s of the Poincaré line bundle on the product $X \times \text{Pic}^0(X)$ such that the restriction D_γ to $X \times \{\gamma\}$ of the divisor $D = \sum m_i D_i$ of the section s meets the curve $C \times \{\gamma\}$ properly and this intersection is contained in $C'_j \times \{\gamma\}$. Clearly, this condition holds for all closed points β from a sufficiently small open neighborhood U of the point γ in $\text{Pic}^0(X)$. Let a_{ij} be the degree of the natural finite map $D_i \cap (C_j \times U) \rightarrow U$. We get a regular morphism

$$\tilde{f} : U \rightarrow \prod_i \text{Sym}^{a_{ij}}(C'_j) \xrightarrow{f} \mathbb{G}_m,$$

where $f = \prod_i (\text{Sym}^{a_{ij}}(f_j))^{m_i}$. Moreover, for any point $\beta \in U$, there is an equality $\tilde{f}(\beta) = \prod_{i,j} \prod_{x \in (D_i)_\beta \cap C_j} f_j^{m_i a_{ij}(x)}(x)$, where $a_{ij}(x)$ is the intersection index of the divisor $(D_i)_\beta$ with C_j at a point x . Therefore by Corollary 4.23, $\pi_*(\{f_j\} \cdot \beta) = \tilde{f}(\beta)^{(-1)^d}$. This proves the needed statement. \square

By Proposition 4.26, for any elements $\alpha \in CH^d(X)_l$, $\beta \in \text{Pic}^0(X)_l$, the direct image of the Massey triple product $\overline{m}_3(\alpha, l, \beta) = \pi_*(m_3(\alpha, l, \beta))$ is well defined.

Proposition 4.27. *Suppose that l is prime to $\text{char}(k)$; then for any elements $\alpha \in CH^d(X)_l$, $\beta \in \text{Pic}^0(X)_l$, there is an equality*

$$\overline{m}_3(\alpha, l, \beta) = \psi_l(\alpha, \beta)^{(-1)^d},$$

where ψ_l is the Weil pairing between the l -torsion of Albanese and Picard varieties of X .

Proof. Since both sides of the equality evidently do not change under extensions of the ground field, we can assume that the field k is algebraically closed. First, suppose that $\dim(X) = 1$ and consider elements $\mathcal{L}, \mathcal{M} \in \text{Pic}^0(X)_l$. Choose two adeles (in fact, ideles) $f, g \in \mathbf{A}(X, \mathcal{K}_1^X)^1$ that correspond to \mathcal{L} and \mathcal{M} , respectively. There are two adeles $\tilde{f}, \tilde{g} \in \mathbf{A}(X, \mathcal{K}_1^X)^0$ such that $d\tilde{f} = f^l$, $d\tilde{g} = g^l$ (we write the group law for K_1 -groups in the multiplicative way). By definition, we have

$$\overline{m}_3(\mathcal{L}, l, \mathcal{M}) = \prod_{x \in X} (\tilde{f}_X, g_{Xx})_x (f_{Xx}, \tilde{g}_x)_x,$$

where $(\cdot, \cdot)_x$ is the tame symbol in the discrete valuation ring $\mathcal{O}_{X,x}$. We may assume that the supports of the divisors $D = -\text{div}(f)$ and $E = -\text{div}(g)$ do not intersect. In this case we get $\phi_l(\mathcal{L}, \mathcal{M}) = f_X(E) \cdot g_X(-D)$. This formula for the Weil pairing $\psi_l(\mathcal{L}, \mathcal{M})$ of \mathcal{L} and \mathcal{M} is well known. The proof can be found in [13], [17], and [10] (these three proofs use different methods).

Now suppose that X is not a curve. Let $i : C \hookrightarrow X$ be a general $(d-1)$ -th hyperplane section of X . Consider two elements $\mathcal{M} \in \text{Pic}^0(C)_l$, $\mathcal{L} \in \text{Pic}^0(X)_l$. The projection formula for Weil pairing (following, for example, from that for étale cohomology) implies that $\psi_l(i_*(\mathcal{M}), \mathcal{L}) = \psi_l(\mathcal{M}, i^*(\mathcal{L}))$. The projection formula for Massey higher products, stated in Lemma 4.1, implies that $\phi_l(i_*(\mathcal{M}), \mathcal{L}) = \phi_l(\mathcal{M}, i^*(\mathcal{L}))^{(-1)^{d-1}}$. On the other hand, it is well known that the map $i_* : \text{Pic}^0(C)_l \rightarrow \text{Alb}(X)_l$ is surjective. Thus, by the previous step, we get the needed result. \square

Propositions 4.25 and 4.27 imply the following formula.

Corollary 4.28. *Let the class $\alpha \in CH^d(X)_l$ be represented by a zero-cycle $z = \sum_i m_i \cdot z_i$ and let the class $\beta \in \text{Pic}^0(X)_l$ be represented by a divisor D such that z does not intersect with D . Suppose that $\text{div}(g) = lD$ with $g \in k(X)^*$ and $d(\{f_m\}) = lz$, where d is the differential in the Gersten complex on X , $f_m \in k(C_m)^*$, and C_m are irreducible curves on X . Assume that for any m the rational function f_m is regular at all points from the*

intersection of D with C_m . Then in the notations from Proposition 4.25 we have the following formula for the Weil pairing of α and \mathcal{L} :

$$\psi_l(\alpha, \mathcal{L}) = \left(\prod_{x \in C \cap D} f(x) \cdot \prod_i g^{-m_i}(z_i) \right)^{(-1)^d}.$$

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